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ON THE CANONICAL FORM OF THE SINGULAR MATRIX PENCIL

By A. C. AITKEN (*Edinburgh*)

[Received 31 July 1933]

THE object of this note is to derive Kronecker's canonical form of a singular pencil of matrices by elementary rational methods. The feature of the method is that the requisite transformations of the pencil are studied by way of the reciprocal transformations of certain row and column vectors intimately connected with the pencil.

1. The singular pencil and its minimal vectors

A pencil of matrices, such as

$$\lambda\begin{bmatrix} 1 & \cdot \\ 1 & \cdot \end{bmatrix}+\mu\begin{bmatrix} \cdot & 1 \\ \cdot & 1 \end{bmatrix}=\begin{bmatrix} \lambda & \mu \\ \lambda & \mu \end{bmatrix}, \tag{1}$$

is singular, in that its determinant vanishes identically in λ, μ. In general a matrix pencil $\Lambda = \lambda A+\mu B$ is singular when its determinant $|\lambda A+\mu B|$ is identically zero, or when the matrix $\lambda A+\mu B$ is rectangular, not square.

With respect to example (1) it may be observed that

$$[1, \quad -1]\begin{bmatrix} \lambda & \mu \\ \lambda & \mu \end{bmatrix}=[0, \quad 0]; \tag{2}$$

also that

$$\begin{bmatrix} \lambda & \mu \\ \lambda & \mu \end{bmatrix}\begin{bmatrix} \mu \\ -\lambda \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3}$$

Indeed the quality of being singular consists in a *dependence* existing among the rows or the columns of the pencil, expressible by either or both of the statements

$$\theta\Lambda = 0, \qquad \Lambda\phi = 0, \tag{4}$$

where θ and ϕ are row and column vectors whose elements are polynomials in λ and μ. Since in any element of the vectors $\theta\Lambda$ and $\Lambda\phi$ all terms of each degree must vanish *separately*, it follows that there exist vectors θ and ϕ satisfying (4) and such that

$$\begin{aligned} \theta &= [\theta_1, \theta_2,\ldots, \theta_p], \\ \phi &= \{\phi_1, \phi_2,\ldots, \phi_q\}, \end{aligned} \tag{5}$$

where the elements θ_j and ϕ_i are polynomials homogeneous in λ, μ,

and each of a certain degree, the *same* for all the θ_j, and similarly for the ϕ_i.

Further, as the simple example (1), with (2) and (3), also shows, there is a *minimal* degree for such polynomials, in the sense, for instance, that

$$\begin{bmatrix} \lambda & \mu \\ \lambda & \mu \end{bmatrix}$$

reduces to zero the vector $\{\mu, -\lambda\}$, which has elements of the first degree, but not (identically) any vector of the form $\{1, a\}$, having elements of zero degree.

We shall use the index m for the minimal degree in vectors θ, and m' for that in vectors ϕ; and it will be said that the rows of Λ have dependence of minimal order m in λ, μ, while the columns have dependence of minimal order m'.

These minimal orders have invariant properties (**3**); they are invariant under equivalent non-singular transformations of Λ, and also under non-singular linear transformations of λ, μ to ρ, σ. The proof of these statements is immediate; in either case we have only to suppose the contrary and then to apply the reciprocal (non-singular) transformation, which would yield an identity of less than minimal order, involving an absurdity.

2. Transformation of the minimal vector

The vector
$$\theta = [\theta_1, \theta_2, ..., \theta_p] \tag{6}$$
will now be studied, and transformed. The elements θ_j are polynomials of degree m; hence if there are more than $m+1$ non-zero elements θ_j they cannot be linearly independent. They must be expressible linearly in terms of a basic set of $m+1$ polynomials, perhaps even fewer. We may take as basic set

$$\mu^m,\ -\lambda\mu^{m-1},\ \lambda^2\mu^{m-2}, ...,\ (-)^m\lambda^m, \tag{7}$$

and we shall prove that none of these can be absent.

For suppose the contrary. Suppose, for example, that the θ_j are linearly expressible in terms of m polynomials only, $\lambda^{k+1}\mu^{m-k-1}$ being missing; so that

$$\theta = [\mu^m, -\lambda\mu^{m-1}, \lambda^2\mu^{m-2}, ..., (-)^k\lambda^k\mu^{m-k}, 0, (-)^{k+2}\lambda^{k+2}\mu^{m-k-2}, ...]H,$$
$$|H| \neq 0. \tag{8}$$

Let a typical column of $H\Lambda$ be the column vector

$$\gamma = \{a_1\lambda+b_1\mu, a_2\lambda+b_2\mu, ..., a_p\lambda+b_p\mu\}. \tag{9}$$

Then, since $\theta\gamma = 0$, we have at once

$$b_1 = 0, \quad a_1 = b_2, \quad a_2 = b_3,..., \quad a_k = b_{k+1}, \quad a_{k+1} = 0, \tag{10}$$

so that $[\mu^m, -\lambda\mu^{m-1}, \lambda^2\mu^{m-2},..., (-)^k\lambda^k\mu^{m-k}, 0, 0,..., 0]\gamma = 0,$ (11)

that is, $[\mu^k, -\lambda\mu^{k-1}, \lambda^2\mu^{k-2},..., (-)^k\lambda^k, 0, 0,..., 0]\gamma = 0,$ (12)

and indeed

$$[\mu^k, -\lambda\mu^{k-1}, \lambda^2\mu^{k-2},..., (-)^k\lambda^k, 0, 0,..., 0]H\Lambda = 0, \tag{13}$$

since γ was any column of $H\Lambda$. But this brings us to a minimal vector of order $k < m$, contrary to the hypothesis that m was minimal. Hence the assumption is false, and θ must be of the form

$$\theta = [\mu^m, -\lambda\mu^{m-1},..., (-)^m\lambda^m, 0, 0,..., 0]H. \tag{14}$$

3. Transformation of the pencil

The matrix $H\Lambda$ will now be studied. No element in its first row can contain μ, for the term in μ^{m+1} thus arising in the null vector $\theta H\Lambda$ is impossible, having no other term in μ^{m+1} to cancel it. Hence the first row of $H\Lambda$ is either a null row, in which case $m = 0$, θ being then

$$[1, 0, 0,..., 0],$$

or else this row contains at least one element $a\lambda$. Let such an element be brought by interchange of columns to the leading position $(1, 1)$, let the factor a be divided out from the first column, and let the λ be used in elementary operations upon columns to clear any other λ's from the first row. These operations are equivalent to post-multiplication by a non-singular matrix K_1, yielding at this stage $H\Lambda K_1$, with first row

$$[\lambda, 0, 0,..., 0].$$

Next consider the second row of $H\Lambda K_1$. Under the λ at $(1, 1)$ there must be, for cancelling purposes, a μ at $(2, 1)$ in an element which may be $\mu + c\lambda$. There can be no other μ in row 2, for the $\lambda\mu^m$ thus occurring in $\theta H\Lambda K_1$ could not be cancelled. If there is no λ in this row either, then $m = 1$, for the minimal vector would then be

$$[\mu + c\lambda, -\lambda, 0, 0,..., 0],$$

with $c = 0$, because of the form of θ in (14). If $m = 1$, then $H\Lambda K_1$ is of the form, as far as its first two rows are concerned,

$$\begin{bmatrix} \lambda & . & . & . & . \\ \mu & . & . & . & . \end{bmatrix}.$$

Suppose, however, that there is a λ in row 2. We bring it as before with unit coefficient to the position $(2, 2)$, and use it to clear other

λ's from the second row. We now have

$$\begin{bmatrix} \lambda & . & . & . & . \\ \mu & \lambda & . & . & . \end{bmatrix}.$$

Under the λ at $(2,2)$ there must be, for cancelling purposes, a μ at $(3,2)$ in an element of the form $\mu+c\lambda$, and in the same way as before there can be no other μ in row 3. If there is no λ either, we have

$$HAK_2 = \begin{bmatrix} \lambda & . & . & . & . \\ \mu & \lambda & . & . & . \\ . & \mu+c\lambda & . & . & . \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix},$$

and then $m = 2$, for

$$[\mu(\mu+c\lambda), -\lambda(\mu+c\lambda), \lambda^2, 0, 0,\dots, 0]H\Lambda K_2 = 0,$$

but again in such a case $c = 0$, in view of (14).

Hence at this stage we have

$$HAK_2 = \begin{bmatrix} \lambda & . & . & . & . \\ \mu & \lambda & . & . & . \\ . & \mu & . & . & . \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

If there is a λ in row 3, we bring it to $(3,3)$ and carry out the same routine of elementary operations as before.

The process must come to an end, and so, for a general value m_1 of m, we must have

$$HAK = \begin{bmatrix} L_m & . \\ * & \Lambda_1 \end{bmatrix}, \qquad L_{m_1} = \begin{bmatrix} \lambda & & & \\ \mu & \ddots & & \\ & \ddots & \lambda & \\ & & \mu & \lambda \\ & & & \mu \end{bmatrix}$$

where L_{m_1} is a rectangular matrix of m_1+1 rows and m_1 columns as shown above, the submatrix denoted by the asterisk being not necessarily null.

If there is row-dependence in the submatrix Λ_1 we reduce it in similar fashion, obtaining

$$\begin{bmatrix} L_{m_1} & . & . \\ * & L_{m_2} & . \\ * & * & \Lambda_2 \end{bmatrix}, \qquad m_2 \geqslant m_1,$$

and so we proceed until row-dependence in the Λ_i's is exhausted, At that stage Λ has been transformed into

$$\begin{bmatrix}
L_{m_1} & & & & & \\
* & L_{m_2} & & & & \\
\cdots & \cdots & & & & \\
\cdots & \cdots & \cdots & & & \\
* & * & \cdots & L_{m_r} & & \\
* & * & \cdots & * & \Lambda_r
\end{bmatrix}.$$

If there exists column-dependence in Λ_r we reduce it in quite analogous, but transposed, fashion, obtaining†

$$\begin{bmatrix}
L_{m_1} & & & & & & & & \\
* & L_{m_2} & & & & & & & \\
\cdots & \cdots & & & & & & & \\
\cdots & \cdots & \cdots & & & & & & \\
* & * & \cdots & L_{m_r} & & & & & \\
* & * & \cdots & * & L'_{m_1} & * & \cdots & * & * \\
* & * & \cdots & * & & L'_{m'_2} & \cdots & * & * \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
* & * & \cdots & * & & & & L'_{m'_s} & * \\
* & * & \cdots & * & & & & & \Lambda_{r+s}
\end{bmatrix},$$

where Λ_{r+s} is either square and non-singular or else does not arise at all. In the former case it can be reduced to either of the canonical forms (the 'classical' and the 'rational') appropriate to non-singular pencils of matrices (**3**).

As for the asterisked sub-matrices, such non-zero elements as they may contain can be reduced to zero by simple elementary operations, which have been described in (**3**).

The matrix of type $H\Lambda K$ that thus emerges is Kronecker's canonical form for the singular matrix pencil (**1**), the invariants under equivalent non-singular transformations being the invariant factors (or if preferred the elementary divisors) of the non-singular core Λ_{r+s}, and the two sets of minimal indices m_i and m'_j (**2**).

† L' denotes the transposed matrix of L.

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ON ANALYTIC FUNCTIONS REGULAR IN THE UNIT CIRCLE

By MARY L. CARTWRIGHT (*Cambridge*)

[Received 6 July 1933]

1.1. LET $$f(z) = f(re^{i\theta}) = u(r,\theta)+iv(r,\theta)$$
be regular for $|z| < 1$, and let
$$v(0) = 0. \tag{1.11}$$
Hardy and Littlewood* have proved that if, as $r \to 1$,
$$M_\lambda(r,u) = \left(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |u(r,\theta)|^\lambda\, d\theta\right)^{1/\lambda} = O\big((1-r)^{-\alpha}\big),$$
where $\alpha > 0$, $\lambda > 0$, then
$$M_\lambda(r,v) = O\big((1-r)^{-\alpha}\big),$$
as $r \to 1$. If $\lambda > 1$, the theorem reduces to M. Riesz's theorem; it is then true for $\alpha = 0$, and can be stated in terms of boundary functions. Making $\lambda \to \infty$, we have the comparatively trivial result that, if
$$u = O\big((1-r)^{-\alpha}\big), \tag{1.12}$$
where $\alpha > 0$, as $r \to 1$, then
$$v = O\big((1-r)^{-\alpha}\big). \tag{1.13}$$
This result is not true for $\alpha = 0$.

The problem before us here is to prove similar results depending on hypotheses on $u_+(r,\theta)$, where $u_+ = u$ if $u > 0$, and $u_+ = 0$ otherwise. Carathéodory's theorem shows that
$$|f(re^{i\theta})-f(0)| \leqslant \frac{2r}{R-r}\{A(R)-u(0)\},$$
where $r < R < 1$, and $A(R) = \max\limits_{0\leqslant\theta\leqslant 2\pi} u(R,\theta)$. It follows from (1.11) that $f(0) = u(0)$, and so, if $u \leqslant 0$ for $r \leqslant 1$, we have
$$M(r) = \max_{0\leqslant\theta\leqslant 2\pi} |f(re^{i\theta})| \leqslant \left\{\frac{2r}{R-r}+1\right\}|u(0)|.$$
Choosing $R = \frac{1}{2}(1+r)$, we have
$$M(r) = O\big((1-r)^{-1}\big). \tag{1.14}$$

* G. H. Hardy and J. E. Littlewood, *Journal für Math.* 167 (1931), 405–23.

If $$u_+(r,\theta) = O\big((1-r)^{-\alpha}\big), \tag{1.15}$$

this method gives $$M(r) = O\big((1-r)^{-\alpha-1}\big). \tag{1.16}$$

The function $f(z) = -(1-z)^{-1}$, whose real part is negative, shows that (1.14) cannot be much improved; but a study of special functions leads one to suppose that (1.16) could be replaced by

$$M(r) = O\big((1-r)^{-1}\big) \quad \text{if} \quad \alpha < 1 \tag{1.17}$$

$$M(r) = O\big((1-r)^{-\alpha}\big) \quad \text{if} \quad \alpha > 1. \tag{1.18}$$

This is in fact true.

It should be observed that, since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(r,\theta)\,d\theta = u(0), \tag{1.19}$$

$M_1(r,u_+)$ and $M_1(r,u_-)$, where $u_- = u$ if $u < 0$ and 0 otherwise, are of the same order. Hence theorems about $M_1(r,u_+)$ and $M_1(r,f)$ can be obtained immediately from the theorems of Hardy and Littlewood for conjugate functions. The theorems for $M_\lambda(r,u_+)$, where $1 < \lambda < \infty$, are more difficult. They will be considered in another paper.

1.2. The proof of (1.13) was shown to me by Professor Littlewood; and, as it has not yet been published, I shall begin by reproducing it. I next consider the case in which (1.15) is satisfied for $\alpha < 1$. When this is so, we can afford to lose something in passing from (1.15) to (1.17), and so we can use a simple method. In proving (1.18) we have to use heavier methods involving the theorems of Phragmén and Lindelöf.

I should like to thank Professor Littlewood for suggesting the problem from which these investigations began and for much valuable advice.

2. We shall prove the results in a more precise form than that stated in the introduction. In order to do this we have to fix the value of f at the origin; if the hypothesis is on $|u|$, it only remains to fix $|v(0)|$ which has already been done by (1.11), but if, as in the later theorems, the hypothesis is on u_+, we have to suppose that $u(0) \geqslant 0$, or some such hypothesis. In what follows A is an absolute constant; B, C, D, K may depend on a parameter ζ, say, in which case we write $B(\zeta)$. A, B, C, D preserve their identity throughout a theorem and its proof, but K need not be the same in each place.

THEOREM 1. *If* $$|u(r,\theta)| < A(1-r)^{-\alpha},$$
where $\alpha > 0$, *for* $r < 1$, *then*
$$|v(r,\theta)| < K(\alpha)A(1-r)^{-\alpha}$$
for $r < 1$.

Poisson's formula is
$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u(R,\phi)(R^2-r^2)\,d\phi}{R^2+r^2-2Rr\cos(\phi-\theta)},$$
where $r < R < 1$; and so we have
$$\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{R}{\pi}\int_{-\pi}^{\pi} \frac{u(R,\phi)(R^2-r^2)\sin(\phi-\theta)\,d\phi}{\{R^2+r^2-2Rr\cos(\phi-\theta)\}^2}.$$
We may without loss of generality suppose that $\theta = 0$, and we have
$$\left|\frac{\partial v}{\partial r}\right| \leqslant A(1-R)^{-\alpha}(R^2-r^2)\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\sin\phi|\,|d\phi|}{\{(R-r)^2+4Rr\sin^2\frac{1}{2}\phi\}^2}.$$
Putting $R = \frac{1}{2}(1+r)$, we have
$$\left|\frac{\partial v}{\partial r}\right| < KA(1-r)^{-\alpha+1}\int_0^{\pi} \frac{\phi\,d\phi}{\{(1-r)^2+\phi^2\}^2} < KA(1-r)^{-\alpha-1}\int_0^{\pi/(1-r)} \frac{\phi\,d\phi}{(1+\phi^2)^2}$$
$$< KA(1-r)^{-\alpha-1}\int_0^{\infty} \frac{\phi\,d\phi}{(1+\phi^2)^2} < KA(1-r)^{-\alpha-1}.$$
Integrating, we have
$$|v(r,0)| < K(\alpha)A(1-r)^{-\alpha},$$
which is the required result.

3.1. In the proofs of the two following theorems we subtract a function whose real part is positive so as to obtain a function whose real part is negative in a certain region. We then apply Carathéodory's theorem in that region if possible; if not, we map the region conformally on a more convenient one, and then apply Carathéodory's theorem.

THEOREM 2. *If* $u(0) \geqslant 0$, *and if*
$$u(r,\theta) < A(1-r)^{-\alpha},$$
where $\alpha < 1$, *for* $r < 1$, *then*
$$M(r) < K(\alpha)A(1-r)^{-1}$$
for $r < 1$.

We first observe that, applying Carathéodory's theorem to $f(z)$ in the circle $|z| < R < 1$, we have

$$|f(re^{i\theta})-f(0)| \leqslant \frac{2r}{R-r}\{A(R)-u(0)\}.$$

Putting $R = \frac{1}{2}(1+r)$, and remembering that $v(0) = 0$, $u(0) \geqslant 0$, we have

$$|f(re^{i\theta})| \leqslant 4A(1-r)^{-\alpha-1}+|f(0)| \leqslant 4A(1-r)^{-\alpha-1}+A. \quad (3.11)$$

It is sufficient to consider a typical point, $z = re^{i\theta_0}$, and show that $|f(re^{i\theta_0})| < K(\alpha)(1-r)^{-1}$, where $K(\alpha)$ is independent of θ_0. There is no loss of generality in supposing $\theta_0 = 0$. Let

$$f_1(z) = u_1+iv_1 = f(z)-KA(1-z)^{-\beta},$$

where $\alpha < \beta < 1$, and consider the value of u_1 inside the region

$$|1-z| \leqslant 2^{\frac{1}{2}}(1-r)^{\alpha/\beta}, \qquad 0 \leqslant |1-z| \leqslant R_0. \quad (3.12)$$

Writing $1-z = \rho e^{i\phi}$, we have

$$\mathbf{R}\{(1-z)^{-\beta}\} = \rho^{-\beta}\cos\beta\phi > \rho^{-\beta}\cos\tfrac{1}{2}\beta\pi$$

inside the unit circle, and so

$$\mathbf{R}\{(1-z)^{-\beta}\} > 2^{-\frac{1}{2}\beta}(1-r)^{-\alpha}\cos\tfrac{1}{2}\beta\pi$$

in the region defined by (3.12). Choosing $K \geqslant 2^{\frac{1}{2}\beta}\sec\frac{1}{2}\beta\pi$, we have

$$\begin{aligned} u_1 &< u-KA\mathbf{R}(1-z)^{-\beta} \\ &< A(1-r)^{-\alpha}-A(1-r)^{-\alpha} = 0 \end{aligned} \quad (3.13)$$

in (3.12).

If $\alpha < \frac{1}{2}$, we can choose $\beta > 2\alpha$, and then it is easy to see that the region (3.12) includes the circle $|z-\frac{1}{2}| \leqslant \frac{1}{2}$, provided that $R_0 > 1$. For inside this circle $r \leqslant \cos\theta$, and so

$$|1-z|^2 = 1-2r\cos\theta+r^2 \leqslant 1-r^2 \leqslant 2(1-r).$$

In this case, applying Carathéodory's theorem to $f_1(z)$ in the circle $|z-\frac{1}{2}| \leqslant \frac{1}{2}$, we have

$$|f_1(r)| \leqslant \frac{|u_1(\frac{1}{2})|}{1-r}+|f_1(\tfrac{1}{2})|;$$

and it follows from (3.11) that

$$|u_1(\tfrac{1}{2})| \leqslant |f_1(\tfrac{1}{2})| < (2^{-\alpha+3}+1)A+K(\beta)A\,2^{\beta} < K(\alpha,\beta)A.$$

Hence

$$\begin{aligned} |f(r)| &\leqslant |f_1(r)|+K(\beta)A|1-re^{i\theta}|^{-1} \\ &< K(\beta)A(1-r)^{-1} = K(\alpha)A(1-r)^{-1}, \end{aligned}$$

which is the required result.

If $\alpha > \frac{1}{2}$, we put

$$1-z = (1-w)\{1-(1-w)^\gamma\}, \tag{3.14}$$

where $0 < \gamma < 1$, $1-z = \rho e^{i\phi}$, $1-w = \sigma e^{i\psi}$. Then for appropriate R_0, R_0', γ, the region defined by

$$|\psi| \leqslant \tfrac{1}{2}\pi, \qquad \sigma \leqslant R_0' \tag{3.15}$$

is included in that defined by (3.12), viz.

$$\rho \leqslant 2^{\frac{1}{2}}(1-r)^{\alpha/\beta}, \qquad \rho \leqslant R_0.$$

For, as $\rho \to 0$, $\sigma \sim \rho$, and $\phi = \psi - \chi$, where $\tan\chi = \dfrac{\sigma^\gamma \sin\gamma\psi}{1-\sigma^\gamma\cos\psi}$, so

$$\chi \sim \rho^\gamma \sin\gamma\psi.$$

Hence $|\psi| \leqslant \frac{1}{2}\pi$ is practically

$$-\tfrac{1}{2}\pi \leqslant \phi + \rho^\gamma \sin\gamma\psi \leqslant \tfrac{1}{2}\pi.$$

If ϕ is near $\frac{1}{2}\pi$, so is ψ, and if ϕ is near $-\frac{1}{2}\pi$, so is ψ. It follows that the region (3.15) is included in

$$\tfrac{1}{2}\pi - |\phi| \geqslant \tfrac{1}{2}\rho^\gamma \sin\tfrac{1}{2}\gamma\pi, \qquad \rho \leqslant R_0 \tag{3.16}$$

for sufficiently small R_0'. But, since $\theta < 2\rho$, as $z \to 1$, we have

$$\begin{aligned}\tfrac{1}{2}\pi - |\phi| < 2\cos\phi = 2(1-r\cos\theta)|1-z|^{-1} &< 2(1-r+\tfrac{1}{2}r\theta^2)|1-z|^{-1} \\ &< 2(1-r)\rho^{-1} + 2r\rho \end{aligned} \tag{3.17}$$

for sufficiently small ρ. It follows that the region (3.16) is included in

$$\tfrac{1}{2}\rho^\gamma \sin\tfrac{1}{2}\gamma\pi < 2(1-r)\rho^{-1} + 2r\rho, \qquad \rho \leqslant R_0,$$

i.e. $$\rho^{\gamma+1} - 2\rho^2 \operatorname{cosec}\tfrac{1}{2}\gamma\pi \leqslant 4(1-r)\operatorname{cosec}\tfrac{1}{2}\gamma\pi, \qquad \rho \leqslant R_0.$$

Since $\gamma < 1$, this region is included in

$$\rho^{\gamma+1} < 8(1-r)\operatorname{cosec}\tfrac{1}{2}\gamma\pi, \qquad \rho \leqslant R_0,$$

and, if $\gamma+1 < \beta/\alpha$, this region is included in (3.12) for sufficiently small R_0'.

Now we apply Carathéodory's theorem to

$$\Phi(w) = f_1\{w + (1-w)^{\gamma+1}\}$$

in the circle $|w - 1 + \frac{1}{2}R_0'| \leqslant \frac{1}{2}R_0'$. It follows from (3.13) that $\mathbf{R}\{\Phi(w)\} \leqslant 0$ in this circle, and so

$$|\Phi(|w|)| \leqslant \frac{2\mathbf{R}\{\Phi(1+\frac{1}{2}R_0')\}}{1-|w|} + |\Phi(1+\tfrac{1}{2}R_0')|.$$

Since R_0' depends only on α, β, γ, it follows from (3.11) that

$$|\Phi(1+\tfrac{1}{2}R_0')| < K(\alpha,\beta,\gamma)A = K(\alpha,\beta)A;$$

and, since $|1-w| \sim |1-z|$, we have

$$|f_1(r)| < K(\alpha,\beta)A(1-r)^{-1},$$

from which the required result follows as before.

3.2. For $\alpha > 1$, the method gives the following result, which will be superseded by Theorem III: *If* $u(0) \geqslant 0$, *and if*

$$u(r,\theta) < A(1-r)^{-\alpha},$$

where $\alpha > 1$, *then, given any* $\epsilon > 0$, *we have*

$$M(r) < A(1-r)^{-\alpha-\epsilon} \tag{3.21}$$

provided that $r_0(\epsilon) < r < 1$.

We have

$$\mathbf{R}\{(1-z)^{-\alpha}\} \geqslant \cos\alpha\beta|1-z|^{-\alpha} \geqslant \cos\alpha\beta(\cos\beta)^{\alpha}(1-r)^{-\alpha}$$

for $|\arg(1-z)| \leqslant \beta < \frac{1}{2}\pi/\alpha$, $|1-z| < R_0 = R_0(\beta)$. Hence

$$\mathbf{R}\{f_1(z)\} = \mathbf{R}\{f(z)-A\sec\alpha\beta(\sec\beta)^{\alpha}(1-z)^{-\alpha}\} \leqslant 0$$

in that region. Putting $1-z = (1-w)^{2\beta/\pi}$, we have

$$\mathbf{R}\{\Phi(w)\} = \mathbf{R}\{f_1(1-(1-w)^{2\beta/\pi})\} \leqslant 0$$

in the circle $|w-1+\frac{1}{2}R_0'| \leqslant \frac{1}{2}R_0'$ for sufficiently small R_0'. Applying Carathéodory's theorem to $\Phi(w)$, we have

$$|\Phi(|w|)| \leqslant \frac{2\mathbf{R}\{\Phi(1+\frac{1}{2}R_0')\}}{1-|w|}+\Phi(1+\tfrac{1}{2}R_0').$$

But R_0' depends only on β, and (3.11) still holds. Hence

$$|\Phi(|w|)| < K(\beta)A(1-|w|)^{-1},$$

and so

$$\begin{aligned}|f(r)| &\leqslant |f_1(r)|+A\sec\alpha\beta(\sec\beta)^{\alpha}(1-r)^{-\alpha}\\ &< K(\beta)A(1-r)^{-2\beta/\pi}\end{aligned}$$

for $R_0'' < r < 1$. Given any $\epsilon > 0$, we can choose β so near to $\frac{1}{2}\pi/\alpha$ that $2\beta/\pi < \alpha+\frac{1}{2}\epsilon$, and then R_0'' so near 1 that $(1-r)^{-\frac{1}{2}\epsilon} > K(\beta)$ for $R_0'' < r < 1$, and we have the required result.

4.1. It remains to prove the more precise results for $\alpha \geqslant 1$.

THEOREM III. *If* $u(0) \geqslant 0$, *and if*

$$u(r,\theta) < A(1-r)^{-\alpha},$$

where $\alpha > 1$, *then*

$$u(r,\theta) > -K(\alpha)A(1-r)^{-\alpha},$$

and so

$$M(r) < K(\alpha)A(1-r)^{-\alpha}.$$

This theorem is an immediate corollary of the following lemma.

LEMMA 1. *Suppose that $F(z)$ is regular and*

$$\log|F(re^{i\theta})| < Br^{\alpha}$$

for $r > l$, $|\theta| \leqslant \beta$, where $\beta > \frac{1}{2}\pi/\alpha$. Suppose also that $F(z)$ has no zeros in that region. Then, given any $\delta > 0$, we can choose $K(\delta)$ so that*

$$\log|F(re^{i\theta})| > -K(\delta)Br^{\alpha}$$

for $|\theta| \leqslant \beta - \delta$, $r > l$.

For if we consider the typical point $z = 1$, and put $1 - z = 1/w$, we find that, since $\alpha > 1$,

$$F(w) = \exp\{f(1-1/w)\}$$

satisfies the hypotheses of the lemma for some β such that $\frac{1}{2}\pi/\alpha < \beta < \frac{1}{2}\pi$ with $l = \sec\beta$, and $B = K(\beta)A$. Hence, putting $\delta = \frac{1}{2}\beta$, we have

$$u = \log|F(w)| > -K(\beta)A|w|^{\alpha}$$
$$= -K(\alpha)A|1-z|^{-\alpha} > -K(\alpha)A(1-r)^{-\alpha}$$

on the real axis, which is the required result.

4.2. The lemma depends chiefly on the theorems of Phragmén and Lindelöf.† If $B \to 0$ as $l \to \infty$, the lemma reduces to a simplified form of a theorem which I have given elsewhere,‡ and the proof for $B > 0$ follows the same lines. The fundamental theorem of Phragmén and Lindelöf on which nearly all the others depend may be stated as follows:

THEOREM OF PHRAGMÉN AND LINDELÖF. *Suppose that $F(z)$ is regular and*

$$\log|F(re^{i\theta})| < Br^{\alpha}$$

for $|\theta| \leqslant \beta < \frac{1}{2}\pi/\alpha$, $r > l$, and let

$$H(\theta) = \overline{\lim_{r\to\infty}} \frac{\log|F(re^{i\theta})|}{r^{\alpha}}.$$

Suppose further that

$$H(\theta) \leqslant C\cos\theta\alpha + D\sin\theta\alpha$$

for $\theta = \pm\beta$; then, given any $\epsilon > 0$, we have

$$\log|F(re^{i\theta})| < \{H(\theta)+\epsilon\}r^{\alpha} \leqslant (C\cos\theta\alpha + D\sin\theta\alpha + \epsilon)r^{\alpha}$$

for $|\theta| \leqslant \beta$ and every $r > r_0(\epsilon)$.

* If $F(z)$ is not identically zero, but has some zeros in the sector, it is still possible to obtain a result of this kind which is valid except in certain circles near the zeros. Compare M. L. Cartwright *Proc. London Math. Soc.* (unpublished).

† E. Phragmén and E. Lindelöf, *Acta Math.* 31 (1908), 386.

‡ M. L. Cartwright, *Proc. London Math. Soc.* (unpublished).

Proof of Lemma 1. We may, without loss of generality, suppose that $\beta < \pi/\alpha$. For if not, we can obtain the required result by dividing the angle $|\theta| \leqslant \beta$ into a number of angles each less than $2\pi/\alpha$ and applying the result for $\beta < \pi/\alpha$ in each of them. We have $H(\theta) \leqslant B$. Suppose first that $H(\gamma) = -\infty$, where $|\gamma| \leqslant \beta$. Then, applying the above theorem in the angles $\gamma_1 \leqslant \theta \leqslant \gamma, \gamma \leqslant \theta \leqslant \gamma_2$, where $-\beta \leqslant \gamma_1$, $\gamma-\gamma_1 < \pi/\alpha$, $\gamma_2-\gamma < \pi/\alpha$, $\gamma_2 \leqslant \beta$, we see that $H(\theta) = -\infty$ for $\gamma_1 < \theta < \gamma_2$. And, repeating the argument if necessary so as to cover the whole angle, we have $H(\theta) = -\infty$ for $|\theta| < \beta$. Since $\beta > \pi/\alpha$, it follows* that $F(z) \equiv 0$, which is contrary to our hypotheses. Hence $H(\theta)$ is finite for $|\theta| \leqslant \beta$.

In this case, since $\frac{1}{2}\pi/\alpha < \beta < \pi/\alpha$, we have†

$$H(0) \geqslant \tfrac{1}{2}\{H(\beta)+H(-\beta)\}\sec\beta\alpha.$$

For if
$$H(0) = \tfrac{1}{2}\{H(\beta)+H(-\beta)\}\sec\beta\alpha-\zeta, \tag{4.21}$$
where $0 < \zeta < \infty$, we have

$$H(\theta) \leqslant \tfrac{1}{2}\{H(\beta)+H(-\beta)\}\sec\beta\alpha\cos\theta\alpha+ \\ +\tfrac{1}{2}\{H(\beta)-H(-\beta)\}\operatorname{cosec}\beta\alpha\sin\theta\alpha+\zeta\operatorname{cosec}\beta\alpha\sin(\theta-\beta)\alpha \tag{4.22}$$

for $\theta = 0$ and $\theta = \beta$. Hence (4.22) holds for $0 \leqslant \theta \leqslant \beta$. Similarly,

$$H(\theta) \leqslant \tfrac{1}{2}\{H(\beta)+H(-\beta)\}\sec\beta\alpha\cos\theta\alpha+ \\ +\tfrac{1}{2}\{H(\beta)-H(-\beta)\}\operatorname{cosec}\beta\alpha\sin\theta\alpha-\zeta\operatorname{cosec}\beta\alpha\sin(\beta+\theta)\alpha \tag{4.23}$$

for $-\beta \leqslant \theta \leqslant 0$. But it follows from (4.22) and (4.23) that

$$H(\theta) \leqslant \tfrac{1}{2}\{H(\beta)+H(-\beta)\}\sec\beta\alpha\cos\theta\alpha+ \\ +\tfrac{1}{2}\{H(\beta)-H(-\beta)\}\operatorname{cosec}\beta\alpha\sin\theta\alpha- \\ -\zeta\operatorname{cosec}\beta\alpha\sin(\beta-\gamma)\alpha\sec\gamma\alpha\cos\theta\alpha, \tag{4.24}$$

where $0 < \gamma < \frac{1}{2}\pi/\alpha$, for $\theta = \pm\gamma$. Hence (4.24) holds for $|\theta| \leqslant \gamma$, which contradicts (4.21) if γ is chosen equal to $\beta-\frac{1}{2}\pi/\alpha$.

Consider the function

$$\Phi(z) = F(z)\exp\{(C+iD+\zeta)z^{\alpha}\}\exp(-\xi r_0^{\alpha}),$$

where $\zeta > 0$, $\xi > 0$,

$$C = -\tfrac{1}{2}\{H(\beta_1)+H(-\beta_1)\}\sec\beta_1\alpha,$$
$$D = \tfrac{1}{2}\{H(\beta_1)-H(-\beta_1)\}\operatorname{cosec}\beta_1\alpha,$$

and $\frac{1}{2}\pi/\alpha < \beta_1 < \beta$. Now, given any $\epsilon_1 > 0$, we can choose $r_0 = r_0(\epsilon_1)$ so that

$$\log|F(re^{i\theta})| < \{H(\theta)+\epsilon_1\}r^{\alpha}$$

* See Pólya, and Szegö, *Aufgaben und Lehrsätze*, i (1925), 148, No. 327.

† Compare G. Pólya, *Math. Zeitschrift*, 29 (1929), 574.

for $|\theta| \leqslant \beta_1$, $r > r_0(\epsilon_1)$; and so, if $\epsilon_1 < -\zeta\cos\beta_1\alpha$, we have

$$\log|\Phi(re^{i\beta_1})| < [H(\beta_1)+\epsilon_1-\tfrac{1}{2}\{H(\beta_1)+H(-\beta_1)\}-$$
$$-\tfrac{1}{2}\{H(\beta_1)-H(-\beta_1)\}+\zeta\cos\beta_1\alpha]r^\alpha < (\epsilon_1+\zeta\cos\beta_1\alpha)r^\alpha < 0$$

for $r > r_0$, and similarly $\log|\Phi(re^{-i\beta_1})| \leqslant 0$ for $r > r_0$. We choose ξ so that

$$\log|\Phi(r_0 e^{i\theta})| < 0$$

for $|\theta| \leqslant \beta_1$.

It follows from the definition of $H(\theta)$ that, given any $\epsilon_2 > 0$, we can find a sequence $R_1, R_2,..., R_m \to \infty$ such that

$$\log|F(R_m)| > \{H(0)-\epsilon_2\}R_m^\alpha \geqslant [\tfrac{1}{2}\{H(\beta_1)+H(-\beta_1)\}\sec\beta_1\alpha-\epsilon_2]R_m^\alpha.$$

Hence
$$\log|\Phi(R_m)| > (\zeta-\epsilon_2)R_m^\alpha-\xi r_0^\alpha > 0,$$

provided that $\epsilon_2 < \zeta$ and $m > m_0(\epsilon_1, \epsilon_2, \xi)$. We apply the maximum modulus principle to $\Phi(z)$ in the region $r_0 \leqslant r \leqslant R$, $R > R_{m_0}$, $|\theta| \leqslant \beta_1$. Since $|\Phi(z)| < 1$ for $|z| = r_0$, $|\theta| \leqslant \beta_1$, and for $\theta = \pm\beta_1$, $r_0 \leqslant r < R$, while $|\Phi(R_m)| > 1$, there must be a point, $Re^{i\Theta}$, on the arc $|z| = R$, $|\theta| \leqslant \beta_1$, at which

$$|\Phi(Re^{i\Theta})| > 1.$$

Taking this point as centre and applying Carathéodory's theorem* to $\log\Phi(z)$ in the circle $|z-Re^{i\Theta}| \leqslant R\sin(\beta-\beta_1)$, we have

$$\log|\Phi(z)| > -K(\beta-\beta_1)BR^\alpha$$

for $|z-Re^{i\Theta}| \leqslant \tfrac{1}{2}R\sin(\beta-\beta_1)$. That is,

$$\log|\Phi(Re^{i\theta})| > -K(\beta-\beta_1)BR^\alpha \tag{4.25}$$

for $|\theta-\Theta| \leqslant \tfrac{1}{2}(\beta-\beta_1)$.

Let Θ_1 and Θ_2 be the two extreme points of the arc

$$|\theta-\Theta| \leqslant \tfrac{1}{2}(\beta-\beta_1)$$

which lie inside the arc $|\theta| \leqslant \beta_1$. One of them may coincide with β_1 or $-\beta_1$, or both may lie inside $|\theta| < \beta_1$. Using (4.25), we apply Carathéodory's theorem to $\Phi(z)$ in the circles

$$|z-Re^{i\Theta_1}| \leqslant R\sin(\beta-\beta_1), \qquad |z-Re^{i\Theta_2}| \leqslant R\sin(\beta-\beta_1),$$

and it follows that (4.25) holds for

$$|\theta-\Theta_1| \leqslant \tfrac{1}{2}(\beta-\beta_1), \qquad |\theta-\Theta_2| \leqslant \tfrac{1}{2}(\beta-\beta_1).$$

Hence (4.25) holds for that part of the arc $|\theta-\Theta| \leqslant \beta-\beta_1$ which lies inside $|\theta| \leqslant \beta_1$. Repeating the process not more than $4\beta_1/(\beta-\beta_1)$

* Perhaps it would be more correct to say that we apply Valiron's theorem to $\Phi(z)$, see G. Valiron, *Lectures on the General Theory of Integral Functions*, Cambridge (1923), 89.

times, we find that (4.25) holds for $|\theta| \leqslant \beta_1$, $R > R_{m_0}$, which is equivalent to the desired result.

4.3. For $\alpha = 1$, we have the following theorem:

THEOREM IV. *If $u(0) \geqslant 0$, and if*
$$u(r,\theta) < A(1-r)^{-1},$$
then
$$M(r) < KA(1-r)^{-1}\left(\log\frac{1}{1-r}\right)^{-2}. \tag{4.31}$$

The number 2 in (4.31) cannot be replaced by any smaller number, as may be seen by considering the function
$$f(z) = -\frac{1}{1-z}\left(\log\frac{1}{1-z}\right)^2.$$
Putting $1-z = \rho e^{i\phi}$, we have
$$u = -\frac{\cos\phi}{\rho}\left\{\left(\log\frac{1}{\rho}\right)^2+\phi^2\right\}+\frac{2\phi\sin\phi}{\rho}\log\frac{1}{\rho}.$$
If $\cos\phi > \pi\{\log(1/\rho)\}^{-1}$, then $u < 0$, and if
$$0 < \cos\phi \leqslant \pi\{\log(1/\rho)\}^{-1},$$
then
$$1-r \leqslant \pi\rho\{\log(1/\rho)\}^{-1}.$$
Hence
$$u \leqslant (1-r)^{-1}$$
for $r < 1$. But obviously
$$M(r) \sim \frac{1}{1-r}\left(\log\frac{1}{1-r}\right)^2$$
as $r \to \infty$.

The theorem is a corollary of the following lemma, which corresponds to Lemma 1.

LEMMA 2. *Suppose that $F(z)$ is regular and*
$$\log|F(re^{i\theta})| < Br$$
for $r > l$, $|\theta| \leqslant \frac{1}{2}\pi$. Suppose also that $F(z)$ has no zeros in that region. Then, given any $\delta > 0$, we can choose $K(\delta)$ so that
$$\log|F(re^{i\theta})| > -K(\delta)Br\log r$$
for $|\theta| \leqslant \frac{1}{2}\pi-\delta$, $r > R_0$.

For considering the typical point $z = 1$, we have
$$u < KA\frac{1}{|1-z|}\log\frac{1}{|1-z|}$$
in the region
$$1-r \geqslant K|1-z|\left\{\log\frac{1}{|1-z|}\right\}^{-1}, \qquad |1-z| \leqslant R_0. \tag{4.32}$$

Putting $1-z = (1-w)\log 1/(1-w)$, $1-z = \rho e^{i\phi}$, $1-w = \sigma e^{i\psi}$, we find that the region (4.32) includes the region

$$|\psi| \leqslant \tfrac{1}{2}\pi, \qquad \sigma \leqslant R_0', \tag{4.33}$$

provided that K and R_0' are suitably chosen. For $\rho \sim \sigma\{\log(1/\sigma)\}$ as $\rho \to 1$, and

$$\phi-\psi = \arg\left\{\log\frac{1}{1-w}\right\} \sim \tan\left\{\psi\left(\log\frac{1}{\sigma}\right)^{-1}\right\}$$

$$\sim -\psi\left(\log\frac{1}{\sigma}\right)^{-1} \sim -\psi\left(\log\frac{1}{\rho}\right)^{-1}.$$

Hence (4.33) is included in

$$|\phi| \leqslant \tfrac{1}{2}\pi - \tfrac{1}{2}\psi(\log 1/\rho)^{-1}, \qquad \rho \leqslant R_0,$$

i.e.

$$\tfrac{1}{2}\pi - |\phi| \geqslant \tfrac{1}{4}\pi(\log 1/\rho)^{-1}, \qquad \rho \leqslant R_0;$$

and so it follows from (3.17) that (4.33) is included in

$$\tfrac{1}{4}\pi(\log 1/\rho)^{-1} \leqslant 2(1-r)\rho^{-1} + 2r\rho, \qquad \rho \leqslant R_0,$$

which is included in (4.32) for $K > \frac{1}{8}\pi$, $R_0 = R_0(K)$.

Hence in the region (4.33)

$$\mathbf{R}\{\Phi(w)\} = \mathbf{R}\left\{f\left(1-(1-w)\log\frac{1}{1-w}\right)\right\} < KA\frac{1}{|1-w|}.$$

It follows that

$$F(w') = \exp\left\{\Phi\left(1-\frac{1}{w'}\right)\right\}$$

satisfies the hypotheses of the lemma for some B, and we have

$$u = \mathbf{R}\{\Phi(w)\} = \log|F(w')| > -KAw'\log w'$$

$$= -KA\frac{1}{|1-w|}\log\frac{1}{|1-w|} \sim -KA\frac{1}{|1-z|}\left(\log\frac{1}{|1-z|}\right)^2$$

for $|\arg(1-z)| \leqslant \frac{1}{2}\pi - \delta$, $|1-z| \leqslant R_0'''$. Hence

$$u(r,0) > -KA\frac{1}{1-r}\left(\log\frac{1}{1-r}\right)^2;$$

and it follows that

$$|v(r,0)| < KA\frac{1}{1-r}\left(\log\frac{1}{1-r}\right)^2$$

by the method used in Theorem I.

4.4. Lemma 2 depends on a simple special case of Carleman's formula* which may be stated as follows: *Suppose that* $F(z)$ *is regular and has no zeros for* $|z| \geqslant l$, $|\arg z| \leqslant \frac{1}{2}\pi$. *Then*

* See G. Pólya and G. Szegö, *Aufgaben und Lehrsätze*, i (1925), 120, No. 178.

$$\frac{1}{\pi R}\int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\log|F(Re^{i\theta})|\cos\theta\,d\theta+$$
$$+\frac{1}{2\pi}\int\limits_{l}^{R}\{\log|F(ir)|+\log|F(-ir)|\}\left(\frac{1}{r^2}-\frac{1}{R^2}\right)dr$$

is bounded as $R\to\infty$.

Proof of Lemma 2. Since $\log|F(\pm ir)|<Br$ for $r>l$, we have

$$\frac{1}{2\pi}\int\limits_{l}^{R}\{\log|F(ir)+\log|F(-ir)|\}\left(\frac{1}{r^2}-\frac{1}{R^2}\right)dr<\frac{B}{\pi}\int\limits_{l}^{R}\frac{dr}{r}<\frac{B}{\pi}\log\frac{R}{l}.$$

Hence $$(1/\pi R)\int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\log|F(Re^{i\theta})|\cos\theta\,d\theta>-(B/\pi)\log R+K_1. \tag{4.41}$$

It follows that, given any $\epsilon>0$, we can choose $K(\epsilon)$ so that

$$\log|F(Re^{i\theta})|>-K(\epsilon)BR\log R, \tag{4.42}$$

except in a set of θ of measure less than ϵ, for $|\theta|\leqslant\frac{1}{2}\pi$, $R>l$. For if $\log|F(Re^{i\theta})|<-KBR\log R$ in a set of θ of measure ϵ, then

$$(1/\pi R)\int\limits_{E}\log|F(Re^{i\theta})|\cos\theta\,d\theta<-(KB/\pi)\log R\int\limits_{0}^{\epsilon}\sin\theta\,d\theta$$
$$<-(KB/\pi)\cos\epsilon\log R, \tag{4.43}$$

where E denotes the set of θ of measure ϵ. The integral taken over the complementary set is certainly less than

$$(B/\pi)\int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\cos\theta\,d\theta=2B/\pi, \tag{4.44}$$

and so, if $K\cos\epsilon>1$, (4.43) and (4.44) give a contradiction of (4.41) for $R>R_0(\epsilon,K)$. Now we apply Carathéodory's theorem to

$$\Phi(z)=\{\log F(z)\}/R\log R,$$

in the circle $|z-Re^{i\Theta_1}|\leqslant R\sin(\frac{1}{2}\pi-\Theta_1)$, where Θ_1 belongs to the set for which (4.42) holds and $|\Theta_1|<\epsilon$. Given any $k>1$, we have

$$\log|\Phi(z)|>-K(\epsilon,k)B$$

for $|z-Re^{i\Theta_1}|\leqslant Rk^{-1}\sin(\frac{1}{2}\pi-\Theta_1)$; and, given any $\delta>0$, we can choose ϵ and k so that this circle includes the arc $|\theta|\leqslant\frac{1}{2}\pi-\delta$, $|z|=R\sin\delta$ for all $R>R_0$. Hence

$$\log|F(Re^{i\theta})|>-K(\epsilon,k)BR\sec\delta\log(R\sec\delta)$$
$$>-K(\delta)BR\log R$$

for $R>R_0$, which is the required result.

THE ENERGETICS OF NON-STEADY STATES, WITH APPLICATION TO CEPHEID VARIATION

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1. THE following paper develops a new method for discussing non-steady states of thermal systems. A thermal system is in a 'steady state' when the temperature at every point, not necessarily uniform, remains constant in time. In that case the loss of energy in each small element in any interval of time due to any causes must be balanced by an equal gain due to other causes. Integrating over the whole system, we have the result that the rate of loss of energy, at the surface, or at internal bounding surfaces delimiting 'foreign' bodies not parts of the given system, must equal the rate of gain of energy from given sources. The simplest example is an electric fire, the source of energy here being the applied voltage; another example is a star in a steady state, where the radiation to space is balanced by internal sources of gravitational or sub-atomic energy. The present paper sets out a general method of solving the question of what happens when a system in a steady state is disturbed, it being assumed that the supply of energy is continued at the same constant rate as before. The theory should apply to Cepheid variable stars. The results show that according to circumstances the system may or may not return to the steady state; it may return to it by 'damped oscillations', or in a 'fully damped' manner, or it may diverge from it, in a systematic direction or by oscillations of increasing amplitude. But the most interesting result is that the disturbed system may under certain conditions oscillate with a definite period, and that in this case the oscillations are definitely of non-harmonic type, possessing maxima of energy-output which are more pronounced than the minima. This asymmetry is a well-known characteristic of the light-curves of Cepheids, which thus receive a measure of explanation.

The theory is concerned solely with energy considerations. Just as in dynamical problems of one degree of freedom, in which the motion can be obtained from the constancy of total energy without consideration of reactions, external or internal, so the present theory fixes attention on the constancy of rate of input of energy and

determines the nature of the output. All consideration of internal circumstances such as pressures, temperatures, etc., is avoided. The systems examined differ from conservative mechanical systems in that the total energy stored in the system is not constant in time but fluctuates.

The analysis hinges on a functional equation of the type

$$\kappa\phi(t)+\phi\{t+b+\phi(t)\} = 0,$$

and the paper has a mathematical interest of its own, apart from its physical applications, in the classification of solutions $\phi(t)$ of this equation according to the values of the constants κ and b. The constant κ plays the part of a damping parameter; its value determines whether the system is oscillatory or non-oscillatory, stable or unstable, and if oscillatory then whether it is periodic. The constant b determines the period in the case of strictly periodic disturbances, and the quasi-period in the case of general oscillatory disturbances. The use of the functional equation may be considered as alternative to that of exponential functions and Fourier analysis, which play here no part. It is remarkable that the above simple functional equation itself generates the classification of disturbances classically treated by the Lagrangian method of using exponentials; its advantage is that it generates periodic functions directly, and not as the sum of a number of harmonics.

2. Consider a system which generates energy, or to which energy is supplied, at the constant rate E. Choose a zero of time. Then at time t after this zero, the total amount of energy generated or supplied is Et.

3. Let $f(t)$ be the total amount of energy lost by the system in the same interval $(0, t)$. It may have been emitted to space, as in the case of a star; or converted into mechanical energy and then back into heat by friction with bounding surfaces, as in the case of 'Griffiths's Heat Engine',* or in other ways. If the state of the system is steady, $f'(t) = E$, and the total energy imprisoned in the system remains constant. By integrating we have

$$f(t) = Et. \tag{1}$$

* E. H. Griffiths, *Thermal Measurement of Energy* (Cambridge, 1901), 50. I am indebted to Professor H. F. Newall for this reference, and for introducing me to this example of the general theory.

In a non-steady state $f(t)$ will not be in general equal to Et. The energy imprisoned in the system, say $\Omega(t)$, will not be constant in time, and by counting up gains and losses we have

$$\Omega(t) = Et - f(t) + \Omega_0, \tag{2}$$

where Ω_0 is the energy imprisoned at $t = 0$.

4. As time goes on, in general $f(t)$ steadily increases. In the case of a star, this is obvious, as the differential coefficient $f'(t)$ is just $L(t)$, the absolute bolometric luminosity, which is always positive. The following arguments do not depend on the assumption that $f(t)$ steadily increases, but it is clear that in the neighbourhood of any *stable* steady state $f(t)$ roughly keeps pace with Et. Corresponding to any epoch t, there will be an epoch $t+T$, where T is a function of t such that $f(t+T)$, the energy emitted up to time $t+T$, differs from Et only by an amount depending on the initial conditions. We write therefore

$$f(t+T) = E(t+c), \tag{3}$$

where c is a constant, depending on the conditions at $t = 0$. It is clear in fact that the early behaviour of $f(t)$, near $t = 0$, may be quite independent of the rate of supply; for the system may be started in a state far removed from the steady state corresponding to E, and its initial radiation to space will depend on this initial state. The physical meaning of c is seen by putting $t = 0$ in (3); we get

$$f(T_0) = Ec,$$

where T_0 is the value of T at $t = 0$. Thus Ec is the emission of energy between $t = 0$ and $t = T_0$. Roughly speaking T_0 is a measure of the time that must elapse before the supply influences the emission, but this is only a crude interpretation. The definition of T is simply (3). T may be called the time-lag.* The idea of a time-lag in Cepheid variation between the generation of energy in the interior and its emergence at the surface is not new; it has been specifically mentioned by Eddington† and by Russell, Dugan, and Stewart.‡ Eddington definitely contemplates, moreover, the possibility of E remaining constant during the variation.

* 'Time-lag' occurs in the theory of elastic fatigue, and has been treated by Volterra by the method of 'functionals'.

† *Internal Constitution of the Stars* (Cambridge, 1926), 202.

‡ *Astronomy* (Grim, 1927), 2, 767.

5. As a particular case, when the state is steady we have, by (1),

$$f(t+T) = E(t+T),$$

so that comparison with (3) gives

$$T = c$$

and T is constant, as is physically obvious. For this case we write $T = T_1$.*

6. The reader must recognize that the definition of T is simply equation (3), and not seek to impose any more intuitive meaning on T. If we could regard elements of energy as maintaining an independent identity during their career from generation or supply in the interior to emission at the surface, T could be regarded as the mean time of transit. But this is not legitimate. We could, of course, attempt to calculate the delay between a sudden temporary increase in E and the peak of the consequent increase in surface emission in an otherwise steady state, but this is not exactly T as defined. As it is, I propose to regard T as having a physical meaning, and as being a function of the state of the system at time t. Equation (3), then, definitely assures that sooner or later all the energy supplied to the system by time t is subsequently disposed of. I proceed now with the consequences of definition (3) and the assumption that T depends on the state at time t.

7. In the general case of a non-steady state, the time-lag T and the imprisoned energy Ω are both functions of t, and we may therefore represent T as a function of Ω. *It is assumed that T is a one-valued function of Ω.* In the case of periodic oscillations, for example, in which Ω passes from Ω_{min} to Ω_{max} and back to Ω_{min} in a period, the assumption means that T is supposed to take the same value at the two epochs at which Ω takes any given value lying between Ω_{min} and Ω_{max}. It does not seem possible to offer any *a priori* physical justification for this assumption, though the assumption itself appears not unreasonable simply as a hypothesis whose consequences are to be investigated. Actually the consequences are of sufficient interest to afford some *a posteriori* justification for making the assumption. Further, the single-valuedness of T as a function of Ω is readily verified in the later analysis describing the resulting physical situation.†

* For further discussion of the meaning of T_1, see § 12 below.

† I am indebted to Professors D. R. Hartell and R. H. Fowler for their discussions on this point.

We do not, however, assume that the system is *stable* in its steady state, but consider the most general small departures from the steady state and investigate their subsequent behaviour. Write, then,

$$\begin{aligned} T &= F(\Omega) \\ &= F\{\Omega_1+(\Omega-\Omega_1)\}, \end{aligned} \tag{4}$$

where Ω_1 is the energy imprisoned in the steady state. Since $\Omega-\Omega_1$ is a small quantity, we have approximately

$$T = F(\Omega_1)+(\Omega-\Omega_1)F'(\Omega_1).$$

Substituting for Ω its expression (2), we have

$$T = F(\Omega_1)+\{Et-f(t)+\Omega_0-\Omega_1\}F'(\Omega_1).$$

Write this as

$$T = \gamma+\frac{\kappa}{E}\{f(t)-Et\}, \tag{5}$$

where γ and κ are constants. Their values are given by

$$\gamma = F(\Omega_1)+(\Omega_0-\Omega_1)F'(\Omega_1),$$

or approximately

$$\begin{aligned} \gamma &= F(\Omega_0) \\ &= T_0, \end{aligned} \tag{6}$$

and by

$$\begin{aligned} \kappa &= -EF'(\Omega_1) \\ &= -E\left(\frac{dT}{d\Omega}\right)_{\Omega=\Omega_1}. \end{aligned} \tag{7}$$

8. We now substitute from (5) in (3), obtaining

$$f\left[t+\gamma+\frac{\kappa}{E}\{f(t)-Et\}\right] = E(t+c). \tag{8}$$

Define a function $\phi(t)$ by the relation

$$f(t) = E\left(a+t+\frac{\phi(t)}{\kappa}\right), \tag{9}$$

a being a constant to be chosen later. Then (8) reduces to

$$a(1+\kappa)+\gamma+\phi(t)+\frac{1}{\kappa}\phi\{t+\gamma+\kappa a+\phi(t)\} = c. \tag{10}$$

Choose a so that

$$a = \frac{c-\gamma}{1+\kappa}. \tag{11}$$

Then

$$\gamma+\kappa a = \frac{\gamma+c\kappa}{1+\kappa}. \tag{12}$$

We put

$$\frac{\gamma+c\kappa}{1+\kappa} = b, \tag{12'}$$

and (10) then becomes

$$\boxed{\kappa\,\phi(t)+\phi\{t+b+\phi(t)\} = 0}\,. \tag{13}$$

This is the fundamental equation which ϕ must satisfy. In terms of ϕ, (5) gives for T

$$T = b+\phi(t). \tag{14}$$

Thus the physical meanings of ϕ are: (a) that it determines the fluctuation of T with time; (b) that it determines the fluctuation with time of the difference between $f(t)$ and its secular term Et. If we write $f'(t) = L(t)$ (adopting the usual astrophysical notation), we have from (9)

$$L(t) = E\left(1+\frac{\phi'(t)}{\kappa}\right). \tag{15}$$

Writing
$$\delta L(t) = L(t)-E$$
and
$$E = L_1,$$
where L_1 is the steady-state luminosity, (15) gives

$$\frac{\delta L(t)}{L_1} = \frac{\phi'(t)}{\kappa}. \tag{16}$$

$\phi(t)$ has the dimensions of a time, and κ is dimensionless.

9. From physical considerations $\phi(t)$ is a continuous differentiable function of t in any finite range of t for $t > 0$. The behaviour of ϕ depends on κ, and we classify the cases as follows:

Case 1. $\kappa = 1$. Equation (13) becomes

$$\phi(t)+\phi\{t+b+\phi(t)\} = 0. \tag{17}$$

Replacing t by $t+b+\phi(t)$, we find

$$\phi(t+2b) = \phi(t). \tag{18}$$

Hence ϕ *is periodic, of period* $P = 2b$. Putting $\kappa = 1$ in (12′), the value of P is $\gamma+c$, or T_0+T_1, so that b is essentially positive.* We shall give yet another physical meaning to b later.†

We next note that $\phi(t)$ cannot take the value $-b$. For if, say, $\phi(t') = -b$, then by (17) $-2b = 0$, which is a contradiction.‡ Since

* It is readily verified that the main result about to be obtained, namely, that the interval between minimum and next succeeding associated maximum of ϕ is less than the interval between the maximum and the close of the period (next associated minimum) holds good for functions ϕ satisfying (17) even if b is a negative number.

† We show that b is the time-average of T.

‡ We exclude the special case $b = 0$ for the time being.

ϕ is continuous and periodic, there must be a range $(-b-\eta, -b+\delta)$ $(\eta, \delta > 0)$ inside which ϕ cannot lie; again, since it is continuous, it must lie wholly on one side of this range or wholly on the other. Hence either $\phi(t) \leqslant -b-\eta$ or $\phi(t) \geqslant -b+\delta$. If $\phi(t) \leqslant -b-\eta$, let it take its upper bound $-b-\eta$ for $t = t'$. Then by (17), putting $t = t'$, $\phi(t'-\eta) = b+\eta$, contradicting $\phi(t) \leqslant -b-\eta$. Hence $\phi(t) \geqslant -b+\delta$. The upper bound of ϕ cannot exceed $b-\delta$. For if, say, $\phi(t') = M > b-\delta$, putting $t = t'$ in (17) we have

$$\phi(t'+b+M) = -M < -b+\delta,$$

contradicting $\phi \geqslant -b+\delta$. The upper bound of ϕ is, in fact, precisely $b-\delta$. For if the lower bound $-b+\delta$ is attained for $t = t_1$, then, by (17), $\phi(t_1+\delta) = b-\delta$, so that the value $b-\delta$ is attained for $t = t_1+\delta$. It follows that

$$-b+\delta \leqslant \phi(t) \leqslant b-\delta,$$

whence

$$0 < \delta \leqslant b.$$

Should $\delta = b$, the solution reduces to $\phi(t) \equiv 0$, and the state is steady.

Thus, provided $\delta \neq b$, every minimum $-b+\delta$ of ϕ is followed by an associated maximum at time δ later, and since $b = \frac{1}{2}P$, δ is less than half the period. Accordingly *the interval from minimum to next succeeding associated maximum is less than half the period*. This associated maximum is easily shown to be followed by an associated minimum at the end of the period (reckoned from the first selected minimum), and accordingly *the interval from maximum to next succeeding associated minimum is greater than half the period*. There may be any number of minima $-b+\delta$ and maxima $b-\delta$ in the period, but they may be associated in pairs in the manner indicated.

The graphs of a typical function ϕ and its derivative are shown below (Fig. 1); ϕ' is here the proportional luminosity fluctuation, $\delta L(t)/L_1$, whilst $-\phi$, apart from a constant, is a measure of the imprisoned energy, since

$$\Omega(t) = -E\{a+\phi(t)\}.$$

The steady-state luminosity L_1 is also the mean value of $L(t)$. For

$$\frac{1}{P}\int_{t_1}^{t_1+P} L(t)\,dt = \frac{1}{P}\int_{t_1}^{t_1+P} E\{1+\phi'(t)\}\,dt = L_1.$$

We have the further quantitative relations

$$\int_{t_1}^{t_1+\delta} \frac{\delta L(t)}{L_1}\, dt = \phi(t_1+\delta)-\phi(t_1) = 2(b-\delta) = P-2\delta,$$

$$\int_{t_1+\delta}^{t_1+P} \frac{\delta L(t)}{L_1}\, dt = \phi(t_1+2b)-\phi(t_1+\delta) = -2(b-\delta) = -(P-2\delta).$$

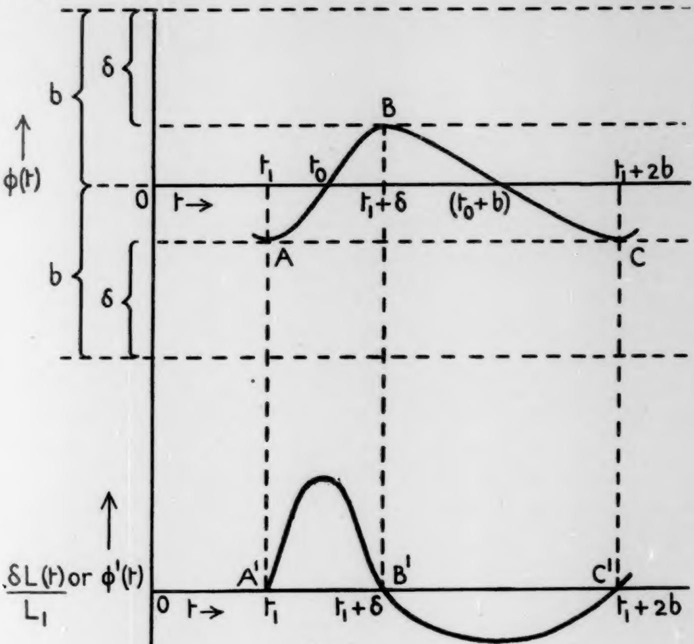

Fig. 1

Hence in the portion of the period, from zero to zero, in which $\delta L(t)$ is on the average positive,

$$\frac{\overline{\delta L(t)}}{L_1} = \frac{P-2\delta}{\delta}; \tag{19}$$

and in the portion of the period, from zero to zero, in which $\delta L(t)$ is on the whole negative,

$$\frac{\overline{|\delta L(t)|}}{L_1} = \frac{P-2\delta}{P-\delta}. \tag{20}$$

These are capable of observational tests on Cepheid variables. In terms of apparent bolometric stellar magnitudes m we have

$$L_1 = A10^{-0\cdot 4m_1}, \qquad L = A10^{-0\cdot 4m},$$

where A is a constant, and m_1 is defined by

$$10^{-0\cdot 4m_1} = \frac{1}{P}\int\limits_{t_1}^{t_1+P} 10^{-0\cdot 4m}\,dt.$$

Relations (19) and (20) then give

$$\frac{1}{\delta}\int\limits_{t_1}^{t_1+\delta} 10^{-0\cdot 4(m-m_1)}\,dt = \frac{P-\delta}{\delta}, \tag{19'}$$

$$\frac{1}{P-\delta}\int\limits_{t_1+\delta}^{t_1+P} 10^{-0\cdot 4(m-m_1)}\,dt = \frac{\delta}{P-\delta}. \tag{20'}$$

It follows at once from these relations that the mean amplitude of the arc of the curve $\delta L(t)$ which rests on the shorter base $A'B'$ (see Fig. 1) is greater than the mean amplitude of the arc which rests on the longer base $B'C'$. This refers to curves on an absolute energy-scale. It is well known that the *light-curves* of Cepheids on a magnitude scale usually show more pronounced maxima than minima, in general agreement with the present theoretical predictions. It may be mentioned that this feature is to be distinguished from another feature of Cepheid light-curves, namely, the steeper rise to maximum than descent from maximum, which is not accounted for by the present theory.

Let t_0 be a zero of $\phi(t)$. Then by (17), putting $t = t_0$, we have $\phi(t_0+b) = 0$. Thus the interval between two associated zeros of ϕ is half the period. It follows that

$$\int\limits_{t_1}^{t_0} \frac{\delta L(t)}{L_1}\,dt = \int\limits_{t_0}^{t_1+\delta} \frac{\delta L(t)}{L_1}\,dt = \tfrac{1}{2}P-\delta \tag{21}$$

$$\int\limits_{t_1+\delta}^{t_0+\frac{1}{2}P} \frac{\delta L(t)}{L_1}\,dt = \int\limits_{t_0+\frac{1}{2}P}^{t_1+P} \frac{\delta L(t)}{L_1}\,dt = -(\tfrac{1}{2}P-\delta). \tag{22}$$

These are also capable of observational test; t_0 is the abscissa of the vertical which divides the lobe $A'B'$ of the energy-curve into two

equal areas; this abscissa should differ by $\frac{1}{2}P$ from that of the vertical which divides the lobe $B'C'$ into equal areas.

The mean value of $\phi(t)$ over a period is zero. For

$$\int_{t_1}^{t_1+2b} \phi(t)\,dt = \int_{t_1}^{t_1+\delta} \phi(t)\,dt + \int_{t_1+\delta}^{t_1+2b} \phi(t)\,dt$$

$$= -\int_{t_1}^{t_1+\delta} \phi\{t+b+\phi(t)\}\,dt + \int_{t_1+\delta}^{t_1+2b} \phi(t)\,dt.$$

In the first of the integrals on the right-hand side, change the variable from t to θ, where

$$t+b+\phi(t) = \theta, \qquad \{1+\phi'(t)\}\,dt = d\theta.$$

By differentiating the functional equation for ϕ, we find at once

$$\{1+\phi'(t)\}\{1+\phi'(\theta)\} = 1.$$

This relation shows that $1+\phi'(t)$ can only change sign if $1+\phi'(\theta)$ becomes infinite, which is excluded on physical grounds. Hence $1+\phi'(t)$ is always positive or always negative, and since it is unity at a maximum or minimum of ϕ, it is always positive. The transformation of variables is therefore legitimate.* The limits for θ are $t_1+\delta$ and t_1+2b. Hence

$$\int_{t_1}^{t_1+2b} \phi(t)\,dt = -\int_{t_1+\delta}^{t_1+2b} \phi(\theta)\{1+\phi'(\theta)\}\,d\theta + \int_{t_1+\delta}^{t_1+2b} \phi(t)\,dt$$

$$= -\tfrac{1}{2}[\phi(\theta)]^2{}_{\theta=t_1+\delta}^{\theta=t_1+2b}$$

$$= 0,$$

since ϕ takes numerically equal values at the limits. It then follows from (14) that $\overline{T} = b$, i.e. the period P is twice the mean value of the time-lag. There is a sort of resonance between the oscillations and the time-lag.

The functional equation in effect maps the arc BC of Fig. 1 on the arc AB. For

as t varies from	t_1	to	$t_1+\delta$,
$\phi(t)$ varies from	$-b+\delta$	to	$b-\delta$,
$t+b+\phi(t)$ varies from	$t_1+\delta$	to	t_1+2b,
$\phi\{t+b+\phi(t)\}$ varies from	$b-\delta$	to	$-b+\delta$.

* I owe this point to Professor Titchmarsh.

In $t_1 < t < t_1+\delta$, we may choose for $\phi(t)$ any function satisfying the conditions

$$\phi(t_1) = -b+\delta, \qquad \phi(t_1+\delta) = b-\delta,$$

$$\phi'(t_1) = \phi'(t_1+\delta) = 0,$$

and otherwise arbitrary; in $t_1+\delta < t < t_1+2b$, $\phi(t)$ is then defined by

$$\phi(t) = -\phi(t') \qquad (t_1 < t' < t_1+\delta),$$

where t' is the appropriate root of

$$t'+b+\phi(t') = t.$$

Thus discussion of the energetics of the system, though fixing certain periodicity properties, leaves a considerable degree of adjustment open for the details of the particular system considered.

It should be noted that by applying the inequality $-b < \phi(t) < b$ to the particular case $t = 0$, at which, by (9), $\phi(0) = -a$, we have

$$-(\gamma+c) < -(c-\gamma) < \gamma+c,$$

which requires $\gamma > 0$, $c > 0$. These conditions are consistent with the physical meanings of c and γ, namely, $c = T_1$, $\gamma = T_0$. They also show why it was necessary to introduce c.

The case $\kappa = 1$ is the most interesting one physically. We proceed now to

Case 2. $0 < \kappa < 1$. By (13), the function $\phi(t)$ takes values of opposite signs. Hence it possesses at least one zero, say $t = t_0$. Then $\phi(t_0) = 0$, and accordingly, by (13),

$$\phi(t_0+b) = 0.$$

Hence
$$\phi(t_0+2b) = 0,$$

and so on. Thus ϕ possesses an unending sequence of zeros at intervals b. (It may have more than one such sequence.) As t varies from t_0 to t_0+b, $\phi(t)$ varies from 0 to 0, and $t+b+\phi(t)$ varies from t_0+b to t_0+2b. Now $t+b+\phi(t)$ is a steadily increasing function, for, using the variable θ again, here $\{1+\phi'(t)\}\{\kappa+\phi'(\theta)\} = \kappa$ and so $1+\phi'(t)$ is always positive.* Hence as t varies from t_0+b to t_0+2b, ϕ passes through the same sequence of values distorted in abscissa and reduced in ordinate in the ratio $-\kappa:1$. The behaviour of ϕ is therefore damped oscillatory, with quasi-period $2b$, and $\phi(t) \to 0$ as $t \to \infty$. Repeated application of the relation $\phi'(\theta) = -\kappa\phi'(t)/\{1+\phi'(t)\}$ shows that $\phi'(t) \to 0$ as $t \to \infty$, and hence $L(t) \to E$. The system is

* Argument due to Titchmarsh.

therefore *stable*, the disturbances die out, and the system returns to the steady state. Ultimately the total energy emitted, namely, $f(t)$, exceeds that generated, namely, Et, by $Ea = E(c-\gamma)/(1+\kappa)$. This depends on the initial conditions, as is to be expected.

Case 3. $\kappa > 1$. The argument given above shows that $\phi(t)$ has an unending sequence of zeros at intervals b, and that the graph of ϕ in any interval between zeros is similar to that in the preceding interval, distorted in abscissa and increased (numerically) in ordinate in the ratio $-\kappa:1$. The oscillations of ϕ therefore increase in amplitude without limit, and the system is *unstable*. Eddington has called this state of affairs 'over-stability'; the system always tends to return to its steady state, but overshoots the mark with continually increasing violence. Of course in this case the physical derivation of (13) ultimately ceases to be valid, but we are now considering (13) on its own merits. This is the usual procedure in questions of stability.

Case 4. $\kappa = 0$. Here $T = \gamma =$ constant, and the state is *steady*.

Case 5. $0 > \kappa > -1$. Put $\kappa = -k$, $0 < k < 1$. Then

$$\phi\{t+b+\phi(t)\} = k\phi(t). \tag{13'}$$

Either ϕ takes the value zero for some value of its argument or it does not. If it does, let t_0 be a zero, $\phi(t_0) = 0$. Then ϕ possesses an unending sequence of zeros at intervals b, and the graph of ϕ in any interval between associated zeros is a distorted copy of that in the preceding interval, with ordinates reduced in the ratio $k:1$. Hence $\phi(t) \to 0$ as $t \to \infty$ and the system is *stable*; the disturbances are damped oscillatory, with quasi-period b (not $2b$).

If ϕ does not take the value zero, it is always of one sign. Suppose (i) that $\phi(t) > 0$ and let $\phi(t') = m > 0$. Then $\phi(t'+b+m) = km$. As t varies from t' to $t'+b+m$, $\phi(t)$ varies from m to km, and $t+b+\phi(t)$ varies from $t'+b+m$ to $t'+2b+m(1+k)$; hence, as t varies from $t'+b+m$ to $t'+2b+m(1+k)$, $\phi(t)$ varies from km to k^2m. Repeating the process, we see that $\phi(t) \to 0$ as $t \to \infty$, and the system is *stable*; the disturbances are non-oscillatory, i.e. aperiodic or 'fully damped'.

Suppose (ii) that $\phi(t) < 0$ and write $\phi(t) = -\psi(t)$. The functional equation becomes

$$\psi\{t+b-\psi(t)\} = k\psi(t). \tag{13''}$$

Now $\psi(t)$ cannot take the value b. For if so $b = kb$, requiring $b = 0$ or $k = 1$, both of which are excluded. Hence either always $\psi(t) > b$ or $\psi(t) < b$. If $\psi(t) > b$, let $\psi(t') = b+\eta'$ $(\eta' > 0)$. Then, by

successive application of (13″),

$$\psi(t'-\eta') = k(b+\eta'),$$
$$\psi\{t'+b(1-k)-\eta'(1+k)\} = k^2(b+\eta'),$$
$$\psi\{t'+b(2-k-k^2)-\eta'(1+k+k^2)\} = k^3(b+\eta'),$$

and so on. We find eventually an argument of ψ_1 necessarily greater than t', for which ψ is as small as we please, contradicting $\psi(t) > b$. Hence $\psi(t) < b$. Now let $\psi(t') = m < b$. As t varies from t' to $t'+b-m$, $\psi(t)$ varies from m to km, and $t+b-\psi(t)$ varies from $t'+b-m$ to $t'+2b-m(1+k)$; hence, as t varies from $t'+b-m$ to $t'+2b-m(1+k)$, $\psi(t')$ varies from km to k^2m. Since $m < b$, the arguments of ψ for the successive intervals so constructed tend to infinity, and $\psi \to 0$. The system is therefore *stable*, and the disturbances again aperiodic or 'fully damped'.

Case 6. $\kappa = -1$. Here (11) gives no determination of a, and the reduction to (13) breaks down. We return to this later.

Case 7. $\kappa < -1$. Put $\kappa = -k$, $k > 1$. The equation is (13′) above. If $\phi(t)$ can take the value zero, a sequence of zeros at intervals b exists, and the usual argument shows that in each interval the graph of ϕ is a distorted copy of that in the preceding interval, increased in the ratio $k:1$. The system is therefore *unstable* (overstable) and the disturbances oscillatory with quasi-period $2b$. If ϕ never takes the value zero, it is wholly positive or wholly negative. If ϕ is always positive, the amplitudes of ϕ in the successive intervals constructed in the usual manner steadily increase; the system is *unstable*, and the disturbances non-oscillatory. If ϕ is always negative, the usual procedure gives no result as the arguments of ϕ ultimately become negative. Thus the usual procedure does not predict the 'future' behaviour of ϕ. It is clear, however, that ϕ cannot tend to a finite limit as $t \to \infty$; and if it were bounded for, say, $t > N$, then by starting with a value of t sufficiently large ($t > N' > N$) we could determine arguments of $\phi(t)$ exceeding N for which ϕ was outside the supposed bounds. Hence $\phi(t)$ cannot be bounded as $t \to \infty$, the system is *unstable*, and the disturbances non-oscillatory.

We return to *Case* 6, $\kappa = -1$. Equation (10) reduces to

$$\phi\{t+\gamma-a+\phi(t)\} = \phi(t)+\gamma-c.$$

Choose $a = \gamma$. Then

$$\phi\{t+\phi(t)\} = \phi(t)+\gamma-c.$$

If $\gamma \neq c$, $\phi(t)$ cannot tend to a limit as $t \to \infty$; and it can be shown that it cannot be bounded. The system is *unstable*, and as $\phi(t)$ cannot take the value zero, the disturbances are non-oscillatory.

10. We have assumed $b \neq 0$, on physical grounds. It seems worth while, however, to complete the mathematical discussion by a brief consideration of the case $b = 0$.

Case 8. $b = 0$, $-1 < \kappa < 1$. The equation is

$$\kappa\,\phi(t) + \phi\{t+\phi(t)\} = 0. \qquad (13^{\text{III}})$$

If $\phi(t') = m$,

$$\phi(t'+m) = -\kappa m$$
$$\phi\{t'+m(1-\kappa)\} = \kappa^2 m$$
$$\phi\{t'+m(1-\kappa+\kappa^2)\} = -\kappa^3 m,$$

and so on. Proceeding to the limit, we have

$$\phi\left(t'+\frac{\phi(t')}{1+\kappa}\right) = 0$$

for arbitrary t'. Hence either $\phi(t) \equiv 0$ or

$$t+\frac{\phi(t)}{1+\kappa} = \text{constant}.$$

If $\phi \equiv 0$, the state is *steady*. If the alternative holds, $\phi'(t)$ is a constant, and the system has a constant luminosity $L(t) = -E/\kappa$, which is not equal to E. The system therefore steadily gains or loses energy.

Case 9. $b = 0$, $\kappa > 1$ or $\kappa < -1$. $\phi(t)$ oscillates with increasing amplitudes or tends to infinity. The system is *unstable*.

Case 10. $b = 0$, $\kappa = 1$. We have already seen from *Case* 1 that as $b \to 0$, the amplitudes tend to zero, the periodic graph being compressed between two approaching horizontals. Thus, if $\kappa = 1$, as $b \to 0$, $\phi(t) \to 0$ for all t. However, when actually $b = 0$, the equation becomes

$$\phi\{t+\phi(t)\} = -\phi(t), \qquad (13^{\text{IV}})$$

and new solutions spring into existence. If $\phi(t') = m$, $\phi(t'+m) = -m$ and $\phi(t'+m)-\phi(t') = -2m$, so that the slope of the chord joining $\{t'+m, \phi(t'+m)\}$ to $\{t', \phi(t')\}$ is constant and equal to -2. It can be shown that in general the functional equation does not suffice to fix the 'future' behaviour of ϕ beyond a certain point. But an interesting particular case is the exact solution $\phi(t) = -2t+C$, which gives, however, $\phi'(t) = -2$, $L(t) = E(1-2) = -E$, a case physically impossible. The important point for our purpose is that (13^{IV}) does not yield periodic solutions.

Case 11. $b = 0$, $\kappa = -1$. This is equivalent to the sub-case $\gamma = c$ of *Case* 6. The equation is now

$$\phi\{t+\phi(t)\} = \phi(t). \tag{13$^{\text{v}}$}$$

Obviously $\phi(t) \equiv$ constant is a solution. If ϕ is not constant, let it take the values A, B, ($A > 0$, $B > 0$, $A \neq B$) for values t', t'' of t. Then

$$\phi(t'+A) = A, \quad \phi(t'+2A) = A, \quad ..., \quad \phi(t'+mA) = A,$$
$$\phi(t''+B) = B, \quad \phi(t''+2B) = B, \quad ..., \quad \phi(t''+nB) = B.$$

Now clearly A and B may be chosen so that they are not linearly dependent,* and then it is possible to find integers m, n so that

$$|(t''+nB)-(t'+mA)| < \epsilon, \qquad |\phi(t''+nB)-\phi(t'+mA)| = |A-B|,$$

where ϵ is arbitrarily small. Now† if $\theta = t+\phi(t)$,

$$\phi'(\theta) = \frac{\phi'(t)}{1+\phi'(t)}.$$

Hence ϕ' cannot take the value -1. But if always $\phi' < -1$, then $\phi'(\theta) > 0$, a contradiction. Hence always $\phi' > -1$ and hence $\phi'(\theta) = 1-\{1+\phi'(t)\}^{-1} < 1$. Now θ passes through all large enough values. For otherwise $|\theta| <$ constant, $|\phi(\theta)| <$ constant, and so $|\phi(t)| <$ constant, or θ is not bounded, which is a contradiction. Hence for all t not too small, $-1 < \phi'(t) < 1$, and so

$$|\phi(t_2)-\phi(t_1)| = \left|\int\limits_{t_1}^{t_2} \phi'(t)\,dt\right| < |t_2-t_1|.$$

Choose $t_2 = t''+nB$, $t_1 = t'+mA$, then

$$|\phi(t_2)-\phi(t_1)| = |A-B| < \epsilon,$$

a contradiction. Hence, if ϕ takes positive values, it can only take one positive value, and so from its continuity it is a constant.

When ϕ takes only negative values, the argument is more difficult. I am indebted to Professor Titchmarsh for the following discussion of this case. Put $\phi(t) = -\psi(t)$, $\psi(t) \geqslant 0$, $t \geqslant 0$. The equation becomes

$$\psi\{t-\psi(t)\} = \psi(t). \tag{13$^{\text{vi}}$}$$

Case (i): $\psi(t) \equiv 0$ is a solution. Case (ii): if $\psi(t) > t$ for all t, $\psi\{t-\psi(t)\}$ is not defined, and the equation furnishes no further information

* i.e. no integers p, q can be found so that $pA-qB = 0$.

† The inequality now to be obtained is due to Titchmarsh.

about ψ. The system is *unstable*. Case (iii): Suppose there is a value of t, say $t_1(>0)$, such that $0 < \psi(t_1) \leqslant t_1$. Let $t_2 = t_1 - \psi(t_1)$. Then $\psi(t_2) = \psi(t_1)$. If $t_2 \geqslant \psi(t_2) = \psi(t_1)$, let $t_3 = t_2 - \psi(t_2)$. Then

$$\psi(t_3) = \psi(t_2) = \psi(t_1),$$

and so on. We ultimately arrive at a t_n such that $t_n < \psi(t_1)$. Then the graph of ψ lies partly above and partly below $\psi(t) = t$, and hence there exists a t_0 such that $\psi(t_0) = t_0$, whence by (13^{VI}) $\psi(0) = t_0$. Since ψ is single-valued, there is only one t_0. The graph therefore passes from $(0, t_0)$ to (t_0, t_0) and for $t > t_0$ lies below $\psi(t) = t$. It is clear that any value taken by $\psi(t)$ anywhere is also taken in $(0, t_0)$. Hence $\psi(t)$ is bounded. By considering the intersections of the graph of $\psi(t)$ with the lines $\psi = t/n$, Titchmarsh now shows that these intersections only occur where $\psi = t_0$, and hence that, for $nt_0 \leqslant t \leqslant (n+1)t_0$, $n(n+1)^{-1}t_0 \leqslant \psi(t) \leqslant (1+n^{-1})t_0$. In particular, $\psi(t) \to t_0$ as $t \to \infty$. It can then be shown that as t_1 varies from nt_0 to $(n+1)t_0$, t_2 varies from $(n-1)t_0$ to nt_0, t_3 from $(n-2)t_0$ to $(n-1)t_0$, and so on. Moreover, since $dt_2/dt_1 = 1-\psi'(t)$, and since $\psi'(t)$, by a modification of an argument previously used, can be shown to be everywhere < 1, $t_2, t_3, \ldots$ increase through the intervals mentioned. Hence $\psi(t)$ passes through the same values in each of the intervals $\{nt_0, (n+1)t_0\}$, and since $\psi(t) \to t_0$ as $t \to \infty$, $\psi(t) = t_0$ always. Hence $L(t) = E$, and the state is *steady*.

11. We have now shown that the functional equation (13) generates of itself, without detailed assumptions about the nature of the system under discussion, all the different dynamically possible types of disturbance, and that the only case in which strictly periodic solutions occur is $\kappa = 1, b \neq 0$. That is, if the supply of energy is at a constant rate E, periodic fluctuations in the emission can only occur, by (7), when

$$\left(\frac{dT}{d\Omega}\right)_{\Omega=\Omega_1} = -\frac{1}{E}. \tag{23}$$

12. Application to Cepheid variation

It follows that a star can only* be a Cepheid when very special conditions are satisfied. In general, for a star which exists, κ will not happen to be unity, and disturbances will rapidly damp out. For an

* Assuming that the rate of generation of energy is unaffected by disturbances and that T is a single-valued function of Ω.

ordinary star, then, $-1 < \kappa < 1$, or

$$-1 < -E\left(\frac{dT}{d\Omega}\right)_{\Omega=\Omega_1} < 1; \tag{24}$$

the central term of the inequality is a sort of damping parameter (dimensionless) which determines the rapidity with which the oscillations die away. When the damping parameter nears the value -1, the star begins to be unstable. When it nears the value $+1$, the oscillations tend to become of constant amplitude and strictly periodic, though asymmetric in form. In the limit when the damping parameter is unity, undamped periodic oscillations are possible. If we call this the Cepheid state, it follows that the Cepheid state can continue indefinitely. It has long been a puzzle, (*a*) why all stars are not Cepheids, (*b*) why Cepheid oscillations maintain themselves apparently undamped. Our theory answers both questions. Only in particular circumstances will a star satisfy condition (23), so that the Cepheid state is not in general possible. But if condition (23) is satisfied, strictly periodic undamped oscillations are compatible with a constant rate of generation of energy. Further, the theory predicts that the fluctuations in the energy output at the surface will not be of simple harmonic form, and that maxima will be more pronounced than minima, in agreement with the observed characteristics of Cepheid light-curves.

The quantitative relations (19) and (20), or (19′) and (20′), together with (21) and (22) offer an ample field for further tests of the theory. Whether they are found to be obeyed or not, the analysis suggests that attention could with advantage be fixed on the relations between the asymmetry of the light-curve (on an absolute energy-scale) and the amplitudes of the oscillations measured from the *mean* ordinate. Cepheid light-curves have usually been discussed with reference to the *median* ordinate of the magnitudes. Further, the asymmetry should be measured by our quantity δ,* the interval between the attainment of mean luminosity on the upward branch and that on the downward branch, and not in terms of the interval from maximum to minimum. 'Steepness-period' relations have been found by Ludendorff† and Vernon Robinson,‡ but they are not easily related

* Better, by δ/P. † *Berlin Sitz.* v (1929).

‡ *Harvard Obs. Bull.* (1930), 872 and 880. Cf. K. Lundmark, *Lund Obs. Circular*, No. 7 (1932), 150.

to our predicted relations between mean amplitude and asymmetry of energy-curve.

If relations (17)–(20) are not found to be obeyed by actual Cepheids, the conclusion would be that the pulsations affect the rate of generation.

Our theory shows immediately why the Cepheids form a definite sequence in the Hertzsprung-Russell diagram or any equivalent plot. A star is defined by two independent variables, luminosity L and mass M, assuming a constant mean chemical composition. Thus our damping parameter κ will be a function of L and M, $\kappa = \kappa(L, M)$. The condition $\kappa(L, M) = 1$ defines a locus in the plane of L and M, which should be the Cepheid sequence. All the remaining tenanted portions of the plane should correspond to $-1 < \kappa < 1$. This shows again that Cepheids should be relatively infrequent amongst the stars. Further, the parameter $b = (\gamma+c\kappa)/(1+\kappa)$ will also be a function of L and M. When $\kappa = 1$, $P = 2b$, and between the relations $\kappa(L, M) = 1$, $b(L, M) = \frac{1}{2}P$, we can eliminate M, obtaining a relation $F(L, P) = 0$. This is the well-known period-luminosity relation.

It is quite unknown whether the stars evolve or not. If they do, a Cepheid stage would only be reached when κ secularly increased up to unity; it would be passed when κ began to decrease again. It is clear that though a star would be unstable for a value of κ not lying inside $-1 < \kappa \leqslant 1$, it is impossible to have an actually unstable disturbance occurring except for values of κ infinitesimally greater than unity. For a star could not reach a value of κ not lying inside $-1 < \kappa \leqslant 1$ without passing through unstable states, which is impossible.

A further test of the theory would be a direct physical calculation of $T(t)$ or $\bar{T}$. For we have seen that $\bar{T} = b = \frac{1}{2}P$, so that the observed period of a Cepheid should be twice the mean value of the calculated time-lag. We have seen that $b = \bar{T} = \frac{1}{2}(T_0+T_1)$, so that the mean lag $\bar{T}$ is not necessarily equal to the 'steady-state lag' T_1, though it must be admitted that the 'steady-state lag' T_1 is a somewhat conventional quantity, incapable of direct calculation from its apparent definition. In connexion with this it may be noted that, from (2) and (9), the mean energy imprisoned in the system is

$$\bar{\Omega} = \Omega_0 - E\alpha = \Omega_0 - \tfrac{1}{2}E(c-\gamma) = \Omega_0 - \tfrac{1}{2}E(T_1 - T_0).$$

Now b and $\bar{\Omega}$ must be independent of the choice of origin of time,

whilst T_0 and Ω_0 depend solely on the choice of the origin of time. Hence $\frac{1}{2}(T_0+T_1)$ and $\Omega_0-\frac{1}{2}E(T_1-T_0)$ are independent of the choice of time-origin. It follows that the conventional quantity T_1 associated with a given periodic disturbed motion depends on the choice of time-origin. If we choose the time-origin so that Ω_0 is equal to the steady-state energy Ω_1, then $T_1 = T_0 = \bar{T}$. This is physically satisfactory.

13. Application to 'Griffiths's Heat Engine'

In this apparatus the bend of a U-tube is sealed with mercury, and one arm of the U sealed to a flask containing air, the other arm being open. When the flask is steadily heated, the mercury begins to oscillate, the oscillations being reported to be of very considerable amplitude. The assumptions of our general theory seem to be fulfilled, probably by adjustment of the rate of supply E. It will be noted from (7) that increase of E increases κ numerically, and so shifts the system in the direction of reduced stability. This is physically obvious, for clearly sufficient increase of E will make the engine unstable. It would be highly interesting to test experimentally whether the oscillations are of the asymmetric non-harmonic type predicted by our theory. This seems probable, from Griffiths's own discussion,* for he points out that the average air-pressure in the flask must be greater during the descent of the mercury (on the flask-side of the U) than during the ascent. The energy supplied to the flask is dissipated by radiation and conduction from the tube.

14. Functional equations of the type (13) make their appearance in other physical problems, in particular in the theory of the synchronization of watches by two relatively stationary observers by means of light-signals. It would render the present paper too long to discuss this problem here. The broad point of physical interest in our analysis is the proof that periodic fluctuations of a system are, under certain circumstances, compatible with a constant supply of energy, and that such fluctuations are necessarily not simple harmonic.

Note added in Proof.

Mr. J. Hodgkinson has kindly pointed out that, by use of the method of finite differences,† formal solutions of the equation

$$\kappa\phi(t)+\phi\{t+b+\phi(t)\} = 0$$

* Loc. cit. 51.

† G. Boole, *Calculus of Finite Differences* (2nd edition, 1872), Ch. XV.

may be obtained in the following forms:

(a) $\kappa > 0$; $\qquad \phi(t) = C(x)\kappa^x \sin \pi x,$

where $$t = A(x)+bx-C(x)\frac{\kappa^x \sin \pi x}{1+\kappa};$$

(b) $\kappa < 0$, $\kappa = -k$ $(k \neq 1)$;

$$\phi(t) = C(x)k^x,$$

where $$t = A(x)+bx+C(x)\frac{k^x}{k-1};$$

(c) $\kappa = -1$; $\qquad \phi(t) = C(x),$

where $$t = A(x)+bx+xC(x);$$

where in each case $C(x)$ and $A(x)$ are arbitrary functions of period 1. Many of the properties of $\phi(t)$ obtained above can be derived from these explicit solutions.

The paper was, however, concerned with functions $\phi(t)$ which were single-valued functions of t, and Mr. Hodgkinson's solution does not yield a single-valued function of t unless some restrictions are placed on $C(x)$ and $A(x)$. Consider, for example, the particular case $\kappa = 1$, for which the solution is

$$\phi(t) = C(x)\sin \pi x$$
$$t = A(x)+bx-\tfrac{1}{2}C(x)\sin \pi x.$$

$\phi(t)$ is periodic of period $2b$, and x increases by 2 as t increases by $2b$. But if it ever occurs that dt/dx becomes negative, the graph of $\phi(t)$ will turn backwards and ϕ will cease to be one-valued; for the graph of $\phi(t)$ can be obtained by plotting $C(x)\sin \pi x$ against bx and shifting each ordinate forward a distance $A(x)-\frac{1}{2}C(x)\sin \pi x$. In particular, as t varies from t to $t' = t+b+\phi(t)$, x increases by 1, and accordingly, in order that $\phi(t)$ shall be single-valued, it is necessary that $t(x+1)-t(x)$, the change of t, shall be positive, i.e. that $b+C(x)\sin \pi x$ shall be positive, i.e. that $|C(x)\sin \pi x| < b$ for all x. The inequality $-b < \phi(t) < b$ then follows. If $C(x)$ is capable of such a value $C(x')$ that $C(x')\sin \pi x' = -b$, then $t(x'+1) = t(x')$, but $\phi(t) = -b$, $\phi(t') = +b$, and ϕ is not single-valued.

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MEAN-VALUE THEOREMS AND THE RIEMANN ZETA-FUNCTION

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1. THE Riemann zeta-function $\zeta(s) = \zeta(\sigma+ti)$ satisfies a mean-value theorem of the form

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T |\zeta(\sigma+ti)|^{2k}\,dt = C_k(\sigma) \tag{1}$$

in the following cases:

(I) when k is real and $\sigma > 1$;

(II) when k is a positive integer and $\sigma > \max\left(\frac{1}{2}, 1-\frac{1}{k}\right)$;

(III) when k is a positive integer and $\sigma > \frac{1}{2}$, if the unproved 'Lindelöf hypothesis' concerning $\zeta(s)$ is true.

Of these results, (I) follows (in virtue of the expansion of $\zeta^k(s)$ given below) from a classical theorem of Hadamard, (II) is due to Schnee in the case $k = 1$ and to Hardy and Littlewood in the case $k = 2$ and may be extended to $k = 3, 4,...$ by the methods of the last-named authors (or by a method developed by Carlson for the case $k = 2$), and (III) is due to Hardy and Littlewood.*

The question of the existence of the mean-value (1) for positive non-integral values of k and for a range of values of $\sigma \leqslant 1$ has been considered by Titchmarsh.† He has extended completely the result (III) which depends on the Lindelöf hypothesis, and, as an example of what may be obtained from his methods without unproved hypothesis, he has shown that the mean-value exists for $k = \frac{1}{2}$ and $\sigma > \frac{13}{16}$. In this paper we shall extend the result (II) completely to positive non-integral values of k; thus, in particular, we show that the mean-value (1) exists for $k = \frac{1}{2}$ and all $\sigma > \frac{1}{2}$.

In all these theorems

$$C_k(\sigma) = \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}},$$

* For details and references see Landau (**7**), (ii) 776–8, 806–19, 905–6, and Titchmarsh (**9**), Ch. I (1.23), II, VI. (The references in heavy type are to the list at the end of this paper.)

† Titchmarsh (**8**).

where the coefficients $d_k(n)$ are defined by the following expansion, which is absolutely convergent for $\sigma > 1$:

$$\zeta^k(s) = \prod_p \left(1-\frac{1}{p^s}\right)^{-k} = \prod_p \left(1+\frac{k}{p^s}+\frac{k(k+1)}{2!\,p^{2s}}+\ldots\right) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}.$$

The series defining $C_k(\sigma)$ is convergent for all $\sigma > \frac{1}{2}$ since

$$d_k(n) = O(n^{\epsilon}),$$

for every fixed positive ϵ, when $n \to \infty$.*

The proof of our extension of (II) is based on (I) for $k > 0$, on the special case $k = 2$ of (II) (or rather the weaker theorem that

$$\frac{1}{T}\int_1^T |\zeta(\sigma+ti)|^4\,dt = O(1) \tag{2}$$

as $T \to \infty$, for every fixed $\sigma > \frac{1}{2}$), and on the Bohr-Landau theorem† that, if δ is fixed and positive, the number of zeros of $\zeta(s)$ in the region $\sigma \geqslant \frac{1}{2}+\delta$, $-T \leqslant t \leqslant T$, is $o(T)$ as $T \to \infty$. From these and simple classical properties of $\zeta(s)$ we deduce our theorem by the application of certain theorems of general function theory. These theorems concern mean-values along lines, and are extensions or adaptations of theorems considered previously by Carlson, by Hardy, Ingham, and Pólya, by Gabriel, and by Besicovitch.‡ We state and prove these general theorems in §§ 2 and 3, and apply them to $\zeta(s)$ in §§ 4 and 5.

2. Theorem A. *Suppose that $f(s) = f(\sigma+ti)$ is regular in the half-strip S defined by $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $t \geqslant t_0$, and satisfies a 'Phragmén-Lindelöf condition'*

$$f(s) = O(e^{e^{ht}}), \qquad 0 < h < \frac{\pi}{\sigma_2-\sigma_1},$$

uniformly in S. Suppose that (when $T \to \infty$)

$$\overline{\lim}\, \frac{1}{T^{\alpha_1}}\left(\int_{t_0}^{T} |f(\sigma_1+ti)|^{1/\alpha_1}\,dt\right)^{\alpha_1} \leqslant A_1,$$

$$\overline{\lim}\, \frac{1}{T^{\alpha_2}}\left(\int_{t_0}^{T} |f(\sigma_2+ti)|^{1/\alpha_2}\,dt\right)^{\alpha_2} \leqslant A_2,$$

* This is classical when k is a positive integer, and the general case follows from this since $|d_k(n)| \leqslant d_l(n)$ where l is an integer greater than $|k|$. [Cf. Titchmarsh (**8**).]

† Bohr and Landau (**2**). There is a more precise theorem due to Carlson, but this and its further refinements are not required. [See Titchmarsh (**9**), 3.6.]

‡ Carlson (**4**); Hardy, Ingham, and Pólya (**6**); Gabriel (**5**); Besicovitch (**1**).

where $\alpha_1, \alpha_2, a_1, a_2, A_1, A_2 > 0$. *Let*

$$\vartheta_1 \geqslant 0, \qquad \vartheta_2 \geqslant 0, \qquad \vartheta_1+\vartheta_2 = 1,$$

and let σ, α, a, A *be defined by*

$$\sigma = \vartheta_1\sigma_1+\vartheta_2\sigma_2, \quad \alpha = \vartheta_1\alpha_1+\vartheta_2\alpha_2, \quad a = \vartheta_1 a_1+\vartheta_2 a_2, \quad A = A_1^{\vartheta_1}A_2^{\vartheta_2}.$$

Then
$$\overline{\lim}\,\frac{1}{T^a}\left(\int_{t_0}^{T} |f(\sigma+ti)|^{1/\alpha}\,dt\right)^{\alpha} \leqslant A, \tag{3}$$

and this holds uniformly in ϑ_1, ϑ_2 *in the sense that, given any positive* ϵ, *we can find a* T_0 *independent of* ϑ_1, ϑ_2, *such that the expression following the sign* $\overline{\lim}$ *in* (3) *is less than* $(1+\epsilon)A$ *for all* $T > T_0$ *and all admissible values of* ϑ_1, ϑ_2.

This theorem is an analogue for mean-values of Theorem 2 of Gabriel's paper **5**. It is related to Theorem 8* of Hardy, Ingham, and Pólya's paper **6** as Gabriel's theorem is related to Theorem 7* of that paper, and (apart from minor complications arising from the use of unilateral means) may be deduced from Gabriel's theorem in much the same way as Theorem 8 was deduced from Theorem 7. We begin by proving two lemmas which embody the main ideas of the proof.

LEMMA 1. *Suppose that* $f(s)$ *is regular in the rectangle* R *defined by* $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $t_0 \leqslant t \leqslant T$; *that*

$$\left(\int_{t_0}^{T} |f(\sigma_1+ti)|^{1/\alpha_1}\,dt\right)^{\alpha_1} \leqslant A_1, \qquad \left(\int_{t_0}^{T} |f(\sigma_2+ti)|^{1/\alpha_2}\,dt\right)^{\alpha_2} \leqslant A_2,$$

where α_1, α_2, A_1, $A_2 > 0$; *and that*

$$|f(s)| < \eta \min(A_1, A_2) \tag{4}$$

on the horizontal sides $t = t_0$ *and* $t = T$ *of* R. *Let* ϑ_1, ϑ_2, σ, α, A *be as in Theorem* A. *Then*

$$\left(\int_{t_0}^{T} |f(\sigma+ti)|^{1/\alpha}\,dt\right)^{\alpha} \leqslant \mathrm{H}A, \tag{5}$$

where H *is a function of* σ_1, σ_2, α_1, α_2, η, *which tends to* 1 *when* $\eta \to +0$ (σ_1, σ_2, α_1, α_2 *remaining fixed*).

Denote the left-hand side of (5) by $I(\vartheta_1, \vartheta_2)$ or $I(\vartheta)$. We first consider pairs $\{\vartheta_1, \vartheta_2\}$ of the form $\{r_1/2^n, r_2/2^n\}$, where r_1, r_2, n are integers; such a pair will be said to be of order n if one of the fractions $r_1/2^n$, $r_2/2^n$ (and therefore also the other) is in its lowest terms. Let $n \geqslant 1$,

* Or the corresponding theorems of Carlson's note (**4**) (398).

and suppose we have proved that

$$I(\vartheta) \leqslant \mathrm{H}_{n-1} A$$

for all pairs $\{\vartheta_1, \vartheta_2\}$ of order not exceeding $n-1$. Let

$$\{\vartheta_1, \vartheta_2\} = \{r_1/2^n, r_2/2^n\}$$

be any pair of order n, so that r_1, r_2 are odd. Then the pairs

$$\{\vartheta_1', \vartheta_2'\} = \{(r_1+1)/2^n, (r_2-1)/2^n\}, \qquad \{\vartheta_1'', \vartheta_2''\} = \{(r_1-1)/2^n, (r_2+1)/2^n\}$$

are of order not higher than $n-1$, so that, by our supposition,

$$I(\vartheta') \leqslant \mathrm{H}_{n-1} A', \qquad I(\vartheta'') \leqslant \mathrm{H}_{n-1} A'', \tag{6}$$

where σ', α', A' relate to the pair $\{\vartheta_1', \vartheta_2'\}$, and σ'', α'', A'' to $\{\vartheta_1'', \vartheta_2''\}$. Now, since $\vartheta_1 = \frac{1}{2}\vartheta_1' + \frac{1}{2}\vartheta_1''$ and $\vartheta_2 = \frac{1}{2}\vartheta_2' + \frac{1}{2}\vartheta_2''$, we have

$$\sigma = \tfrac{1}{2}\sigma' + \tfrac{1}{2}\sigma'', \qquad \alpha = \tfrac{1}{2}\alpha' + \tfrac{1}{2}\alpha'', \qquad A = A'^{\frac{1}{2}} A''^{\frac{1}{2}};$$

also $\sigma - \sigma' = \sigma'' - \sigma = (\sigma_2 - \sigma_1)/2^n$. Hence, by a theorem of Gabriel,* applied to the rectangle bounded by the lines of abscissae σ', σ'', and ordinates t_0, T, we have

$$I(\vartheta) \leqslant \left(\int_{K'} |f(s)|^{1/\alpha'}\, |ds|\right)^{\frac{1}{2}\alpha'} \left(\int_{K''} |f(s)|^{1/\alpha''}\, |ds|\right)^{\frac{1}{2}\alpha''}, \tag{7}$$

where K', K'' are the two halves of the boundary of the rectangle which lie to the left and right respectively of the line of abscissa σ. Now, by (6) and (4), the first factor on the right of (7) is less than

$$\begin{aligned} \{(\mathrm{H}_{n-1} A')^{1/\alpha'} &+ 2(\eta A')^{1/\alpha'} (\sigma_2 - \sigma_1)/2^n\}^{\frac{1}{2}\alpha'} \\ &= \{\mathrm{H}_{n-1}^{1/\alpha'} + (\sigma_2 - \sigma_1) 2^{-(n-1)} \eta^{1/\alpha'}\}^{\frac{1}{2}\alpha'} A'^{\frac{1}{2}} \\ &\leqslant \{\mathrm{H}_{n-1}^{1/\gamma} + (\sigma_2 - \sigma_1)^{\alpha'/\gamma} 2^{-(n-1)\alpha'/\gamma} \eta^{1/\gamma}\}^{\frac{1}{2}\gamma} A'^{\frac{1}{2}} \\ &\leqslant (\mathrm{H}_{n-1}^{1/\gamma} + U u^{n-1} \eta^{1/\gamma})^{\frac{1}{2}\gamma} A'^{\frac{1}{2}}, \end{aligned}$$

where

$$\gamma = \max(\alpha_1, \alpha_2), \qquad U = \max_{\alpha = \alpha_1, \alpha_2} (\sigma_2 - \sigma_1)^{\alpha/\gamma}, \qquad u = \max_{\alpha = \alpha_1, \alpha_2} 2^{-\alpha/\gamma};$$

we have used here the inequality $x^\lambda + y^\lambda \leqslant (x+y)^\lambda$ ($x \geqslant 0$, $y \geqslant 0$,

* Gabriel (**5**), Theorem 1. The need for a result of the type of Lemma 1 arises merely from the use of unilateral means. If we were working with means over $(-T, T)$ in Theorem A [which would, of course, suffice for the application to $\zeta(s)$], we could simply quote Gabriel's Theorem 2.

The characteristic feature of Gabriel's Theorem 2 is convexity in *two* variables (σ and α in our notation). Other theorems embodying this feature were proved later by Carleman (**3**), and there is an earlier remark of Carlson asserting without proof a theorem of a similar nature [Carlson (**4**), near the bottom of p. 397, 'pour $\kappa \geqslant 2$ l'expression ν/κ est une fonction convexe des deux variables σ et $1/\kappa$']. The results given by Carlson and Carleman are insufficiently precise for our purpose.

$\lambda \geqslant 1$), with $\lambda = \gamma/\alpha'$. A similar result holds for the second factor on the right of (7). Hence

$$I(\vartheta) \leqslant (\mathrm{H}_{n-1}^{1/\gamma}+Uu^{n-1}\eta^{1/\gamma})^{\gamma}A'^{\frac{1}{2}}A''^{\frac{1}{2}} = \mathrm{H}_n A,$$

where
$$\mathrm{H}_n^{1/\gamma} = \mathrm{H}_{n-1}^{1/\gamma}+Uu^{n-1}\eta^{1/\gamma}.$$

Now for order 0, i.e. for the pairs $\{1, 0\}$ and $\{0, 1\}$, we obviously have $I(\vartheta) \leqslant A$. It follows by induction that, for all pairs $\{\vartheta_1, \vartheta_2\}$ of order not exceeding n, $I(\vartheta) \leqslant \mathrm{H}_n A$, where

$$\mathrm{H}_n^{1/\gamma} = 1+\sum_{\nu=1}^{n} Uu^{\nu-1}\eta^{1/\gamma} < 1+\frac{U}{1-u}\eta^{1/\gamma},$$

since $0 < u < 1$. Hence (5) holds, with

$$\mathrm{H} = \left(1+\frac{U}{1-u}\eta^{1/\gamma}\right)^{\gamma},$$

for all $\{\vartheta_1, \vartheta_2\}$ of the special form considered, and therefore, by continuity, for all $\{\vartheta_1, \vartheta_2\}$. Since H depends only on σ_1, σ_2, α_1, α_2, η, and tends to 1 when $\eta \to +0$, the lemma is proved.

LEMMA 2. *Let $E(t) = E_k(t) = e^{-e^{kt}}$, where $k > 0$. Then*

$$\varlimsup_{T\to\infty} \frac{1}{T^c}\int_{t_0}^{T} \phi(t)\,dt = \varlimsup_{\omega\to\infty} \frac{1}{\omega^c}\int_{t_0}^{\infty} E(t-\omega)\phi(t)\,dt \tag{8}$$

for any non-negative (integrable) $\phi(t)$ and any real c. Further, if $T > \max(t_0, 1)$ and $\omega = T+\log T$, then

$$\frac{1}{T^c}\int_{t_0}^{T} \phi(t)\,dt \leqslant \left(1+\frac{\log T}{T}\right)^c e^{T^{-k}}\cdot\frac{1}{\omega^c}\int_{t_0}^{\infty} E(t-\omega)\phi(t)\,dt.$$

Denote the left- and right-hand sides of (8) by Λ_1 and Λ_2 respectively, so that $0 \leqslant \Lambda_1, \Lambda_2 \leqslant +\infty$ (with obvious conventions about $+\infty$).

We first prove that $\Lambda_1 \geqslant \Lambda_2$. We may suppose that $\Lambda_1 < +\infty$, since otherwise there is nothing to prove. Then we can choose numbers $C > 0$ and $\tau > \max(t_0, 0)$, so that

$$\Phi(T) \equiv \int_{t_0}^{T} \phi(t)\,dt < CT^c \qquad (T > \tau).$$

Since $0 < E(t) < 1$ for all t, and $\phi(t) \geqslant 0$ for $t \geqslant t_0$, we have

$$\int_{t_0}^{\infty} E(t-\omega)\phi(t)\,dt \leqslant \int_{t_0}^{T} \phi(t)\,dt + \int_{T}^{\infty} E(t-\omega)\phi(t)\,dt \qquad (T > t_0).$$

Integrating by parts, and noting that $-E'(t) > 0$ for all t, and $0 \leqslant \Phi(t) < Ct^c$ for $t > \tau$, we see that, if $T > \tau$, the last integral is

$$-E(T-\omega)\Phi(T)-\int_T^\infty E'(t-\omega)\Phi(t)\,dt < -\int_T^\infty E'(t-\omega)Ct^c\,dt$$

$$= -\int_{T-\omega}^\infty E'(u)C(u+\omega)^c\,du = -\int_{T-\omega}^\infty E'(u)C\omega^c\left(\frac{u}{\omega}+1\right)^c du.$$

Hence, if $T \geqslant \omega \geqslant 1$,

$$\frac{1}{\omega^c}\int_{t_0}^\infty E(t-\omega)\phi(t)\,dt < \left(\frac{T}{\omega}\right)^c \frac{1}{T^c}\int_{t_0}^T \phi(t)\,dt - \int_{T-\omega}^\infty E'(u)C(u+1)^{|c|}\,du.$$

Now let T and ω tend continuously to $+\infty$ in such a way that $T/\omega \to 1$ and $T-\omega \to +\infty$; we infer at once that $\Lambda_2 \leqslant \Lambda_1$.

On the other hand, since $E(t)$ is positive and decreasing, and $\phi(t) \geqslant 0$, we have, for $\omega > 0$ and $T > t_0$,

$$\frac{1}{\omega^c}\int_{t_0}^\infty E(t-\omega)\phi(t)\,dt \geqslant \left(\frac{T}{\omega}\right)^c E(T-\omega)\frac{1}{T^c}\int_{t_0}^T \phi(t)\,dt.$$

Making T and ω tend continuously to $+\infty$ so that $T/\omega \to 1$ and $T-\omega \to -\infty$, we deduce that $\Lambda_2 \geqslant \Lambda_1$. And taking, in particular, $\omega = T+\log T$, we obtain the last part of the lemma.*

Proof of Theorem A. Choose k so that

$$h < k < \frac{\pi}{\sigma_2-\sigma_1},$$

and let

$$e(s) = e^{-e^{-ik\{s-\frac{1}{2}(\sigma_1+\sigma_2)\}}}, \qquad f_\omega(s) = e(s-\omega i)f(s) \quad (\omega > 0).$$

In the half-strip S,

$$|e(s)| = e^{-e^{kt}\cos k\{\sigma-\frac{1}{2}(\sigma_1+\sigma_2)\}},$$

and $$1 \geqslant \cos k\{\sigma-\tfrac{1}{2}(\sigma_1+\sigma_2)\} \geqslant \cos \tfrac{1}{2}k(\sigma_2-\sigma_1) > 0,$$

* Lemma 2 is modelled on Hardy, Ingham, and Pólya (**6**), Lemmas B and C (557–9). The latter might be used instead, but Lemma 2 seems a little simpler because it avoids the additional limiting process $m \to \infty$, and also more natural because a function of the type of $E(t)$ must in any case enter in connexion with the Phragmén-Lindelöf condition.

In stating Lemma 2 we have confined ourselves to what is actually required for the application. But we may add that, if $\Lambda_1 = \Lambda_2$ is finite, then the lower limits λ_1 and λ_2 are also equal (as a glance at the proof will show). We thus have a (trivial) Abelian-Tauberian relationship.

since $0 < \frac{1}{2}k(\sigma_2-\sigma_1) < \frac{1}{2}\pi$. Hence we can choose a positive number ρ (depending on σ_1, σ_2, α_1, α_2, k) such that

$$e^{-k\rho} \leqslant \alpha^{-1}\cos k\{\sigma-\tfrac{1}{2}(\sigma_1+\sigma_2)\} \leqslant e^{k\rho}$$

for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ and all α between α_1 and α_2; and then we shall have

$$E(t+\rho) \leqslant |e(s)|^{1/\alpha} \leqslant E(t-\rho),$$

where $E(t) = E_k(t)$ has the same meaning as in Lemma 2. Hence

$$E(t-\omega+\rho)|f(s)|^{1/\alpha} \leqslant |f_\omega(s)|^{1/\alpha} \leqslant E(t-\omega-\rho)|f(s)|^{1/\alpha} \tag{9}$$

for all s in S and all α between α_1 and α_2.

Now by hypothesis

$$\overline{\lim_{T\to\infty}} \frac{1}{T^{a_1/\alpha_1}} \int\limits_{t_0}^{T} |f(\sigma_1+ti)|^{1/\alpha_1}\, dt \leqslant A_1^{1/\alpha_1}.$$

Hence, by Lemma 2 (with $\omega+\rho$ written in place of ω),

$$\overline{\lim_{\omega\to\infty}} \frac{1}{(\omega+\rho)^{a_1/\alpha_1}} \int\limits_{t_0}^{\infty} E(t-\omega-\rho)|f(\sigma_1+ti)|^{1/\alpha_1}\, dt \leqslant A_1^{1/\alpha_1}.$$

This implies, in virtue of the second inequality (9),

$$\overline{\lim_{\omega\to\infty}} \frac{1}{\omega^{a_1}}\left(\int\limits_{t_0}^{\infty} |f_\omega(\sigma_1+ti)|^{1/\alpha_1}\, dt\right)^{\alpha_1} \leqslant A_1$$

[since $(\omega+\rho)/\rho \to 1$]. Thus, if any positive number η is given, we can find an $\omega_1 = \omega_1(\eta)$* such that

$$\left(\int\limits_{t_0}^{\infty} |f_\omega(\sigma_1+ti)|^{1/\alpha_1}\, dt\right)^{\alpha_1} < (1+\eta)A_1\,\omega^{a_1} \qquad (\omega > \omega_1),$$

and similarly an $\omega_2 = \omega_2(\eta)$ such that

$$\left(\int\limits_{t_0}^{\infty} |f_\omega(\sigma_2+ti)|^{1/\alpha_2}\, dt\right)^{\alpha_2} < (1+\eta)A_2\,\omega^{a_2} \qquad (\omega > \omega_2).$$

Now, if M is the maximum of $|f(s)|$ on the segment $t = t_0$, $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, we have, on this segment, $|f_\omega(s)| \leqslant |f(s)| \leqslant M$ and therefore certainly

$$|f_\omega(s)| < \eta \min\{(1+\eta)A_1\,\omega^{a_1}, (1+\eta)A_2\,\omega^{a_2}\}$$

for $\omega > \omega' = \omega'(\eta)$. And for any fixed ω the same inequality must hold on the segment $t = T$, $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, for all sufficiently large T; for

$$f_\omega(s) = O[\exp\{-e^{kt}e^{-k\omega}\cos\tfrac{1}{2}k(\sigma_2-\sigma_1)+e^{ht}\}] = o(1),$$

* We do not indicate explicitly the dependence of ω_1, ω_2,... on 'constants of the problem' such as f, t_0, σ_1, σ_2, α_1, α_2, a_1, a_2, A_1, A_2.

uniformly in S, as $t \to \infty$, since $k > h > 0$ and $e^{-k\omega}\cos \frac{1}{2}k(\sigma_2-\sigma_1) > 0$. Hence, taking $\omega > \max(\omega_1, \omega_2, \omega') = \omega_0$, applying Lemma 1 to $f_\omega(s)$ and the rectangle $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $t_0 \leqslant t \leqslant T$, and making $T \to \infty$, we obtain

$$\left(\int_{t_0}^{\infty} |f_\omega(\sigma+ti)|^{1/\alpha}\, dt\right)^{\alpha} \leqslant \mathrm{H}(1+\eta)A\omega^{a} \qquad (\omega > \omega_0),$$

where σ, α, a, A are as in the enunciation of the theorem, and $\mathrm{H} = \mathrm{H}(\eta) \to 1$ when $\eta \to +0$. Using now the first inequality (9), changing ω to $\omega+\rho$ and dividing through by ω^a, we deduce that, if $\omega > \omega_0$ $(> \omega_0-\rho)$,

$$\frac{1}{\omega^a}\left(\int_{t_0}^{\infty} E(t-\omega)|f(\sigma+ti)|^{1/\alpha}\, dt\right)^{\alpha} \leqslant \mathrm{H}(1+\eta)A\left(1+\frac{\rho}{\omega}\right)^{a}.$$

It now follows from the last part of Lemma 2 (with $c = a/\alpha$) that, if $T > \max(t_0, \omega_0, 1)$ and $\omega = T+\log T$ $(> \omega_0)$, then

$$\frac{1}{T^a}\left(\int_{t_0}^{T} |f(\sigma+ti)|^{1/\alpha}\, dt\right)^{\alpha} \leqslant \mathrm{H}(1+\eta)A\left(1+\frac{\rho}{\omega}\right)^{a}\left(1+\frac{\log T}{T}\right)^{a} e^{\alpha T^{-k}}.$$

Now, if $\gamma = \max(\alpha_1, \alpha_2)$, $g = \max(a_1, a_2)$, the product of the last three factors on the right is

$$< \left(1+\frac{\rho}{T}\right)^{g}\left(1+\frac{\log T}{T}\right)^{g} e^{\gamma T^{-k}} < 1+\eta$$

for $T > T' = T'(\eta)$. Hence, if $T > \max(t_0, \omega_0, 1, T') = T_0 = T_0(\eta)$,

$$\frac{1}{T^a}\left(\int_{t_0}^{T} |f(\sigma+ti)|^{1/\alpha}\, dt\right)^{\alpha} < \mathrm{H}(1+\eta)^2 A.$$

Now, given $\epsilon > 0$, we can choose $\eta = \eta(\epsilon)$ so that $\mathrm{H}(1+\eta)^2 < 1+\epsilon$, and then the corresponding $T_0 = T_0(\eta) = T_0(\epsilon)$ fulfils our requirements.

3. Theorem B. *Suppose that $f(s)$ is regular in the half-strip S defined by $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $t \geqslant t_0$, and satisfies the following conditions:*

(i) *for a given $q > 0$ the limit*

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} |f(\sigma+ti)|^{q}\, dt = \mathfrak{M}_q(\sigma)$$

exists for an infinity of values of σ having a limit point inside the interval $\sigma_1 < \sigma < \sigma_2$;

(ii) *for some* $Q > q$

$$\frac{1}{T}\int_{t_0}^{T} |f(\sigma+ti)|^Q\,dt = O(1),$$

uniformly for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, *when* $T \to \infty$;

(iii) $$N(T) = o(T)$$

as $T \to \infty$, *where* $N(T)$ *denotes the number of zeros of* $f(s)$ *in the rectangle* $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $t_0 \leqslant t \leqslant T$.

Then the limit $\mathfrak{M}_q(\sigma)$ *exists for all* σ *in the interval* $\sigma_1 < \sigma < \sigma_2$, *and uniformly in any interior interval* $\sigma_1 < \sigma_1' \leqslant \sigma \leqslant \sigma_2' < \sigma_2$.

Further $\mathfrak{M}_q(\sigma)$ *is the value for* $s = \sigma$ *of a function* $\mathfrak{M}_q(s) = \mathfrak{M}_q(\sigma+ti)$ *regular in the strip* $\sigma_1 < \sigma < \sigma_2$.

When q is an even integer, no restriction on $N(T)$ is needed, and (ii) may be replaced by the corresponding condition with $Q = q$ (a wider hypothesis in virtue of Hölder's inequality); this (or rather the analogous result for bilateral means) constitutes Theorem 14 of Hardy, Ingham, and Pólya's paper **6**. The result was extended to general q by Besicovitch, subject to the hypothesis (iii) and to a different type of condition in place of (ii).* It is the substitution of (ii) for the corresponding condition of Besicovitch that is important for our application to $\zeta(s)$. The condition (iii) will be satisfied in the application, in virtue of the theorem of Bohr and Landau quoted in § 1. For convenience we give the proof of Theorem B in full, though it

* Besicovitch (**1**), 375. In our notation Besicovitch's condition takes the form

(ii)′ $$\int_{T}^{T+1} |f(\sigma+ti)|^q\,dt < C \qquad (\sigma_1 \leqslant \sigma \leqslant \sigma_2,\ T \geqslant t_0).$$

But Besicovitch considers, in place of our functions f^q, functions which may have singular points of any kind in S, provided that their distribution is subject to a condition of the type (iii) and that (ii)′ is not violated. Thus our theorem neither includes nor is included by his. For the special class of functions considered here, however, the substitution of (ii) for (ii)′ represents a generalization, since (ii)′ in S implies (ii) (and indeed the boundedness of f) in any interior half-strip S_δ of the form $\sigma_1+\delta \leqslant \sigma \leqslant \sigma_2-\delta$, $t \geqslant t_0+\delta$. In fact, if $s^* = \sigma^*+t^*i$ is any point of S_δ, we have, since $|f|^q$ is subharmonic,

$$\pi\delta^2|f(s^*)|^q \leqslant \int_0^{\delta} r\,dr \int_{-\pi}^{\pi} |f(s^*+re^{\theta i})|^q\,d\theta \leqslant \int_{\sigma^*-\delta}^{\sigma^*+\delta} d\sigma \int_{t^*-\delta}^{t^*+\delta} |f(\sigma+ti)|^q\,dt \leqslant 2\delta C,$$

if $\delta \leqslant \frac{1}{2}$ (as we may suppose without loss of generality). This type of argument (which has been employed by various writers) may be used to simplify the proofs, and generalize the results, of Theorems 2, 10, and 11 of Hardy, Ingham, and Pólya (**6**).

is only in connexion with the condition (ii) that there is any point of novelty.

Let the distinct ordinates of zeros of $f(s)$ in the interior of S be $t_1, t_2,\ldots$, where $t_0 < t_1 < t_2 < \ldots$. Take $l > 0$, and denote by $L(T)$ the set of points t of the interval (t_0, T) lying outside all the intervals $(t_n - l, t_n + l)$ $(n = 0, 1, 2,\ldots)$, and by $l(T)$ the remainder of the interval (t_0, T). If $l(T)$ denotes also the measure of the set $l(T)$, we have, by (iii),

$$l(T) = o(T) \tag{10}$$

as $T \to \infty$. Using Hölder's inequality, (ii), and (10), we now obtain, writing $f = f(\sigma+ti)$,

$$\frac{1}{T}\int\limits_{l(T)} |f|^q\,dt \leqslant \left(\frac{1}{T}\int\limits_{l(T)} |f|^Q\,dt\right)^{q/Q}\left(\frac{1}{T}\int\limits_{l(T)} 1^{Q/(Q-q)}\,dt\right)^{(Q-q)/Q}$$

$$\leqslant \left(\frac{1}{T}\int\limits_{t_0}^{T} |f|^Q\,dt\right)^{q/Q}\left(\frac{l(T)}{T}\right)^{(Q-q)/Q} = O(1)\,o(1) = o(1)$$

uniformly for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$. Hence

$$\frac{1}{T}\int\limits_{t_0}^{T} |f(\sigma+ti)|^q\,dt - \frac{1}{T}\int\limits_{L(T)} |f(\sigma+ti)|^q\,dt \to 0, \tag{11}$$

uniformly for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, when $T \to \infty$.

Now the lines $t = t_n$ divide the interior of S into domains in which $f(s)$ is regular and has no zeros. Define a function $g(s)$ in this set of domains by taking a definite branch of $f^{\frac{1}{2}q}(s)$ in each domain, and let $\bar{g}(s)$ be the conjugate function (so that $\bar{g}(s) = \overline{g(\bar{s})}$ and is regular in the images of the above domains in the real axis). Let

$$G(s, T) = G_l(s, T) = \frac{1}{T}\int\limits_{L(T)} g(s+ui)\bar{g}(s-ui)\,du.$$

For fixed $T > \max(t_0, 0)$, $G(s, T)$ is evidently regular in the domain D_l defined by $\sigma_1 < \sigma < \sigma_2$, $|t| < l$. Also

$$|G(s, T)| \leqslant \left(\frac{1}{T}\int\limits_{L(T)} |g(s+ui)|^2\,du\right)^{\frac{1}{2}}\left(\frac{1}{T}\int\limits_{L(T)} |\bar{g}(s-ui)|^2\,du\right)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{T}\int\limits_{t_0}^{T+l} |f(\sigma+vi)|^q\,dv = O\left(\frac{T+l}{T}\right) = O(1),$$

uniformly in D_l, as $T \to \infty$, by (ii) and Hölder's inequality. Further (if σ is real)

$$G(\sigma, T) = \frac{1}{T} \int\limits_{L(T)} |g(\sigma+ui)|^2 \, du = \frac{1}{T} \int\limits_{L(T)} |f(\sigma+ui)|^q \, du, \tag{12}$$

so that, by (i) and (11), $G(s, T)$ tends to a limit, as $T \to \infty$, for an infinity of (real) s having a limit point in D_l. It follows, by Vitali's theorem*, that $G(s, T)$ tends to a limit $G(s) = G_l(s)$ at all points s of D_l, and uniformly in any closed set in D_l; and $G(s)$ is regular in D_l. Hence, by (11) and (12),

$$\frac{1}{T} \int\limits_{t_0}^{T} |f(\sigma+ti)|^q \, dt \to G(\sigma)$$

for $\sigma_1 < \sigma < \sigma_2$, and uniformly in any interior interval. The last relation shows that the values of $G(s)$ on the real axis are independent of l. Since l may be as large as we please, it follows that there is a $G(s)$ regular in the whole strip $\sigma_1 < \sigma < \sigma_2$ taking the values in question on the real axis. This completes the proof.

4. We can now state and prove our main theorem concerning $\zeta(s)$. The symbol $C_{\frac{1}{2}q}(\sigma)$ has been defined in § 1.

Theorem. *Let λ be a fixed positive number such that*

$$\frac{1}{T} \int\limits_{1}^{T} |\zeta(\sigma+ti)|^\lambda \, dt = O(1) \tag{13}$$

as $T \to \infty$, for every (fixed) $\sigma > \frac{1}{2}$. Let σ_0 and q be two fixed numbers satisfying the relations

$$\tfrac{1}{2} < \sigma_0 < 1, \qquad q > 0, \qquad (1-\sigma_0)q < \tfrac{1}{2}\lambda.$$

Then

$$\frac{1}{T} \int\limits_{1}^{T} |\zeta(\sigma+ti)|^q \, dt \to C_{\frac{1}{2}q}(\sigma), \tag{14}$$

uniformly for $\sigma \geqslant \sigma_0$, when $T \to \infty$.

We first apply Theorem A with $f(s) = \zeta(s)$; $t_0 = 1$; $\sigma_1 = \frac{1}{2}+\delta$, $\sigma_2 = 1+\delta$, where $0 < 2\delta < \sigma_0 - \frac{1}{2}$; $a_1 = \alpha_1 = 1/\lambda$; $a_2 = \alpha_2 = \delta$. All the hypotheses are satisfied; the two upper limits involved are finite,

* See, e.g., Titchmarsh (**10**), 168 (5.21). The theorem is there stated for a *sequence* of functions, but the proof given applies also to a set of functions depending on a continuous parameter.

the first in virtue of (13) and the second because $\zeta(s)$ is bounded on the line $\sigma = 1+\delta$. Hence

$$\left(\frac{1}{T}\int_1^T |\zeta(\sigma+ti)|^{1/\alpha}\,dt\right)^\alpha = O(1) \tag{15}$$

uniformly in ϑ_1, ϑ_2, where ϑ_1, ϑ_2, σ, α are as in Theorem A. It follows from Hölder's inequality (since $\sigma_1 < \sigma_0-\delta < \sigma_2$) that (15) will hold uniformly for $\sigma_0-\delta \leqslant \sigma \leqslant \sigma_2$, if α is replaced by any fixed number greater than $\bar{\alpha}$, the maximum of the values of α corresponding to this range of σ. Now, since $\vartheta_1 = (\sigma_2-\sigma)/(\sigma_2-\sigma_1)$ and $\vartheta_2 \leqslant 1$, we have

$$\bar{\alpha} \leqslant \frac{\sigma_2-(\sigma_0-\delta)}{\sigma_2-\sigma_1}\alpha_1+\alpha_2 = \frac{1-\sigma_0+2\delta}{\frac{1}{2}\lambda}+\delta.$$

Since $(1-\sigma_0)/(\frac{1}{2}\lambda) < 1/q$, we can choose $\delta = \delta(\lambda, \sigma_0, q)$ so that $\bar{\alpha} < 1/(q+\delta)$, and we shall then have

$$\left(\frac{1}{T}\int_1^T |\zeta(\sigma+ti)|^{q+\delta}\,dt\right)^{1/(q+\delta)} = O(1) \tag{16}$$

uniformly for $\sigma_0-\delta \leqslant \sigma \leqslant 1+\delta$.

We now apply Theorem B to $\zeta(s)$ and the half-strip

$$\sigma_0-\delta \leqslant \sigma \leqslant 1+\delta, \qquad t \geqslant 1.$$

We know [see § 1, (I)] that (14) holds for $1 < \sigma \leqslant 1+\delta$; thus the condition (i) of Theorem B is satisfied. Condition (ii) is fulfilled in virtue of (16), and condition (iii) by the Bohr-Landau theorem (since $\sigma_0-\delta > \frac{1}{2}$). Hence, when $T \to \infty$, the left-hand side of (14) tends, uniformly for $\sigma_0 \leqslant \sigma \leqslant 1+\frac{1}{2}\delta$, to a limit $\mathfrak{M}_q(\sigma)$, where $\mathfrak{M}_q(s)$ is a regular function of s in the strip $\sigma_0-\delta < \sigma < 1+\delta$. Now, when $s = \sigma$, $1 < \sigma \leqslant 1+\frac{1}{2}\delta$, $\mathfrak{M}_q(s)$ coincides with the function

$$C_{\frac{1}{2}q}(s) = \sum_{n=1}^{\infty} \frac{d_{\frac{1}{2}q}^2(n)}{n^{2s}}, \tag{17}$$

which is also regular for $\sigma_0-\delta < \sigma < 1+\delta$, the Dirichlet's series being (absolutely) convergent in this strip*. Hence $\mathfrak{M}_q(s) = C_{\frac{1}{2}q}(s)$ through-

* This is obvious from the fact that $d_{\frac{1}{2}q}(n) = O(n^\epsilon)$, but may be inferred independently. For if σ_c is the abscissa of convergence of (17), we must certainly have $\sigma_c \leqslant 1$, since the expansion $\sum d_{\frac{1}{2}q}(n)n^{-s}$ defining $d_{\frac{1}{2}q}(n)$ is absolutely convergent for $\sigma > 1$, and so *a fortiori* is (17). And, if $\sigma_0-\delta < \sigma_c \leqslant 1$, then, since $d_{\frac{1}{2}q}^2(n) \geqslant 0$, $C_{\frac{1}{2}q}(s)$ would have a singular point at $s = \sigma_c$ (by a well-known theorem of Landau), and this is incompatible with the existence of the function $\mathfrak{M}_q(s)$.

out the strip, and in particular for $s = \sigma$, $\sigma_0 \leqslant \sigma \leqslant 1+\frac{1}{2}\delta$. We have thus proved that (14) holds uniformly for $\sigma_0 \leqslant \sigma \leqslant 1+\frac{1}{2}\delta$. But it also holds uniformly for $\sigma \geqslant 1+\frac{1}{2}\delta$ (as a glance at the usual proof for $\sigma > 1$ shows). This completes the proof of the theorem.*

5. In the theorem just proved we may take $\lambda = 4$, in virtue of (2). The resulting proposition is equivalent, except for the difference of notation and for the additional assertion about uniformity, to that enunciated in § 1, namely, (II) with the restriction to integral values of k removed.

We may note also that, if the Lindelöf hypothesis is true, then, by (III) of § 1, we may take λ to be an arbitrarily large even integer, and therefore σ_0 arbitrarily near to $\frac{1}{2}$ and q arbitrarily large. We thus obtain an alternative proof of Titchmarsh's extension of (III) to non-integral k. For this we do not, indeed, require the full force of our theorem, for it is evident that a convexity theorem in the two variables σ and α is not needed in this case.

* The second half of the argument embodies the proposition that, if the left-hand side of (14) is bounded for a given q (and an appropriate range of σ), then (14) is true for any smaller (positive) q. A similar result has been obtained independently by Bohr and Jessen in a forthcoming paper. These authors apply their methods to some further problems where the argument of the present paper would break down owing to the absence of any analogue of the Bohr-Landau theorem.

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THE WEIERSTRASS E-FUNCTION IN DIFFERENTIAL METRIC GEOMETRY

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1. In this note it is shown how the function which is known in the calculus of variations as the Weierstrass E-function* is related to the idea of convexity dealt with in two previous notes by the present author.†

In these notes we proved the existence of *ovaloid* hypersurfaces in a space of paths, a hypersurface being described as ovaloid if it bounds a simple and completely convex region (I). Here we show that the regularity condition, $E > 0$, is necessary and sufficient for small spheres, defined in terms of the fundamental integral, to be ovaloid.

The condition $E > 0$ is seen to be a consequence‡ of

$$g = |g_{ij}| \neq 0, \qquad F(x,dx) > 0 \qquad (dx \neq 0), \tag{1.1}$$

where
$$g_{ij}(x,dx) = \frac{1}{2}\frac{\partial^2 F^2}{\partial\, dx^i\, \partial\, dx^j}.$$

The conditions (1.1) also imply

$$g_{ij}(x,dx)\,\delta x^i \delta x^j > 0 \qquad (dx \neq 0,\, \delta x \neq 0).$$

2. Let the fundamental differential invariant,

$$F(x,dx) = F(x^1,\ldots,x^n,dx^1,\ldots,dx^n),$$

be of class 2, homogeneous of the first degree in dx, and let it satisfy the conditions
$$g \neq 0, \qquad F(x,dx) > 0$$

for all values of x in a given region and all non-zero vectors dx. The Weierstrass function is defined as

$$E(x,p,\xi) = F(x,\xi) - F_{p^i}\xi^i,$$

* For an account of this function see, for instance, J. Hadamard, *Leçons sur le calcul des variations* (Paris, 1910), Livre III, ch. II, or O. Bolza, *Vorlesungen über Variationsrechnung* (Leipzig, 1909). Also we refer particularly to a paper by G. A. Bliss, *Trans. American Math. Soc.* 15 (1914), 369–78.

† *Quart. J. of Math.* (Oxford), 3 (1932), 33–42, and 4 (1933), 226–7. These notes will be referred to as I and II respectively.

‡ Cf. Bolza, loc. cit., p. 245. The functions g and F_1 are related by the equation $g = F^{n+1}F_1$.

where
$$F_{p^i} = \frac{\partial F(x,p)}{\partial p^i}.$$

For a fixed value of x

$$E(x,p,\xi) > 0 \qquad (\xi^i \neq \rho p^i), \tag{2.1}$$

if and only if the loci in the ξ*-space given by*

$$F(x,\xi) = r \qquad (r > 0) \tag{2.2}$$

*are ovaloid with respect to the straight lines.**

First observe that these loci are homeomorphic to an $(n-1)$-sphere. For it follows from the above conditions on F that each ray issuing from the origin meets each of them just once. Also, because of the homogeneity in p and ξ of the functions E and F, it is sufficient to prove this for a particular value of r, and for values of p satisfying (2.2) with $\xi = p$.

The locus (2.2) is ovaloid if and only if the tangent hyperplane at each point lies entirely outside the surface. That is to say, if and only if

$$F(x,\xi)-r > 0 \tag{2.3}$$

for every value of ξ satisfying

$$F_{p^i}(\xi^i-p^i) = 0 \qquad (\xi^i \neq p^i), \tag{2.4}$$

p being an arbitrary point on (2.2). But

$$\begin{aligned} F_{p^i}(\xi^i-p^i) &= F_{p^i}\xi^i-r \\ &= F(x,\xi)-r-E(x,p,\xi). \end{aligned}$$

Therefore the conditions (2.3) and (2.4) are equivalent to (2.1), and the theorem is established.

Since $g \neq 0$ the quadratic form in ξ,

$$g_{ij}(x,p)\xi^i\xi^j,$$

is either positive definite or takes on negative values. In the latter case it will be negative for at least one value of ξ satisfying the relation

$$g_{ij}(x,p)p^i\xi^j = 0.$$

For

$$g_{ij}(x,p)p^ip^j > 0,$$

and if a non-degenerate quadratic form,

$$a_{ij}\xi^i\xi^j,$$

is not positive definite, the polar plane of a given point p,

$$a_{ij}p^i\xi^j = 0,$$

* In the two-dimensional case this has been proved by C. Carathéodory, *Math. Annalen*, 62 (1906), 449, § 2.

contains at least one point for which $a_{ij}\xi^i\xi^j$ differs in sign from $a_{ij}p^ip^j$, provided the latter is not zero.*

Therefore another necessary and sufficient condition for the locus (2.2), which is equally well defined by

$$g_{ij}(x,\xi)\xi^i\xi^j = r^2,$$

to be ovaloid is that†

$$g_{ij}(x,p)\xi^i\xi^j > 0 \qquad (p \neq 0,\ \xi \neq 0). \tag{2.5}$$

Hence (2.5) is equivalent to the condition (2.1). Moreover, this in turn is equivalent to the condition

$$Q(x,p,\xi) = F_{p^ip^j}\xi^i\xi^j > 0 \qquad (\xi^i \neq \rho p^i), \tag{2.6}$$

where
$$F_{p^ip^j} = \frac{\partial^2 F(x,p)}{\partial p^i \partial p^j}.$$

For
$$g_{ij} = FF_{p^ip^j} + F_{p^i}F_{p^j}$$

and
$$F_{p^i} = F^{-1}g_{ik}p^k.$$

Hence
$$F_{p^ip^j} = F^{-3}(g_{ij}g_{kl} - g_{ik}g_{jl})p^kp^l,$$

and it follows from the algebraic theory of quadratic forms that the condition (2.6) is equivalent to (2.5), and therefore to‡ (2.1).

The condition (2.5) is a consequence of (1.1). For if the latter are satisfied, the locus (2.2) is closed and the quadratic form (2.5) is positive definite near at least one point p (e.g. a point on (2.2) at which ξ^1 has a maximum). Therefore (2.5) is satisfied since the signature can only vary as $(p^1,\dots,p^n)$ pass through a set of values for which $g = 0$.

3. Just as a positive Riemannian metric determines a Euclidean geometry in the tangent spaces, so the more general metric

$$ds = F(x, dx)$$

determines a metric geometry with the distance given by

$$\delta(\xi,\eta) = F(q, \xi - \eta),$$

in the tangent space at each point q. According to the last section

* This can be seen at once by transforming to coordinates, η, in which the components of the vector p are $(1, 0,\dots, 0)$ and in which

$$a_{ij}\xi^i\xi^j = \pm\eta_1^2 \pm \eta_2^2 \pm \dots \pm \eta_n^2.$$

† See I, § 4.

‡ For $n = 3$ the conditions (2.1) and (2.6) were shown to be equivalent by W. Behagel, *Math. Annalen*, 73 (1913), 596–9. Another proof, and an extension to $n > 3$, appears in the paper by Bliss referred to above.

a differential metric space* which satisfies the condition (1.1) is one which determines a metric with ovaloid spheres in each tangent space, a sphere being the locus of points equidistant from a given point. Thus in each tangent space there is a geometry of the kind first studied by Minkowski.† As in Riemannian geometry, normal coordinates at a point q are a means of setting up a relation of closest contact between the underlying space and the metric tangent space at q. We shall use this relation to carry over our geometrical interpretation of regularity from the tangent space to the underlying space.

The geodesics are given by equations of the form‡

$$\frac{d^2x^i}{ds^2} = H^i\left(x, \frac{dx}{ds}\right), \tag{3.1}$$

where the functions H are homogeneous of the second degree in dx, and the parameter s measures the arc length. That is to say,

$$F\left(x, \frac{dx}{ds}\right) = 1. \tag{3.2}$$

The solutions to (3.1) are given by

$$x^i = \phi^i(q, p, s),$$

where

$$\begin{cases} \phi^i(q, p, 0) = q^i \\ \left(\dfrac{d\phi^i}{ds}\right)_{s=0} = p^i, \end{cases}$$

and because of the homogeneity of H they may be written as

$$x^i = f^i(q, y), \tag{3.3}$$

with

$$y^i = p^i s. \tag{3.4}$$

If s is to be the arc length, p must be a unit vector. That is to say,

$$F(q, p) = 1. \tag{3.5}$$

The equations (3.3) satisfy the conditions

$$\left(\frac{\partial f^i}{\partial y^j}\right) = \delta^i_j$$

* For an account of these geometrical ideas, introduced by P. Finsler, see L. Berwald, *Atti del Congresso Internazionale dei Matematici* (Bologna, 1928) 263–70.

† 'Theorie der konvexen Körper, insbesondere Begründung ihres Oberflächenbegriffs': *Gesammelte Abhandlungen* (Leipzig, 1911), vol. ii.

‡ See M. Mason and G. A. Bliss, *Trans. American Math. Soc.* 9 (1908), 443.

and determine a transformation to normal coordinates y in which the geodesics through q are given by (3.4). The functions H and their derivatives with respect to x and dx are continuous for all values of x in the region of definition and for all non-zero vectors dx. Therefore the transformation to the coordinates y is defined for all values of x near q.

The locus given by (3.4) with $s = r$ and p given by (3.5) will be called the sphere of radius r and centre q. It is given implicitly by the equation

$$F(q, y) = r. \tag{3.7}$$

Let α be a vector tangent to (3.7) at the point $(p^1 r,..., p^n r)$. That is to say,

$$F_{p^i} \alpha^i = 0.$$

Further, let

$$y^i = y^i(t)$$

be the equation of the geodesic through $(p^1 r,..., p^n r)$ in the direction α, t being the linear function of the arc length determined by the conditions

$$\begin{cases} y^i(0) = p^i r \\ \left(\dfrac{dy^i}{dt}\right)_0 = \alpha^i. \end{cases}$$

Then, as in I and II, we have

$$\begin{aligned} F\{q, y(t)\} - r &= \tfrac{1}{2}t^2[F_{\xi^i\xi^j}\alpha^i\alpha^j + F_{\xi^i} H^i(pr, \alpha)]_{\xi = pr} + O(t^3) \\ &= \tfrac{1}{2}t^2\left\{\frac{1}{r} F_{p^i p^j}\alpha^i\alpha^j + F_{p^i} H^i\right\} + O(t^3) \\ &= \tfrac{1}{2}t^2\left\{\frac{1}{r} Q(q, p, \alpha) + F_{p^i} H^i\right\} + O(t^3). \end{aligned}$$

If r is sufficiently small the coefficient of t^2 has the same sign as $Q(q, p, \alpha)$. Therefore there is a positive function $\delta(r)$, such that

$$F\{q, y(t)\} - r$$

has the same sign as $Q(q, p, \alpha)$, provided

$$-\delta(r) < t < \delta(r).$$

Therefore the argument used in I shows that small spheres are ovaloid if the metric is regular.

If the metric is not regular there is a pair of vectors, p and ξ, associated with a point q, such that

$$Q(q, p, \xi) < 0.$$

Since

$$F_{p^i p^j} p^j = 0,$$

it follows that

$$Q(q, p, \xi + \rho p) = Q(q, p, \xi)$$

for any value of ρ. Therefore the vector

$$\alpha^i = \xi^i - p^i F_{p^j} \xi^j / F(q,p)$$

satisfies the conditions

$$\begin{cases} Q(q,p,\alpha) < 0 \\ F_{p^i} \alpha^i = 0. \end{cases}$$

Therefore regularity is necessary as well as sufficient for convexity, and we have the theorem:

In a differential metric space, sufficiently small spheres are ovaloid with respect to the geodesics if and only if the metric is regular.

INTEGRALS FOR THE PRODUCT OF TWO BESSEL FUNCTIONS (II)

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1.1. The present paper contains results complementary to the formulae obtained in our previous paper* on the same topic. Particular examples of our new results occur in an earlier investigation.†

We find several integral expressions for the products $K_\mu(X)K_\nu(x)$, $K_\mu(X)I_\nu(x)$, $I_\mu(X)K_\nu(x)$; the most noteworthy are those given in 5 (5) and 6 (3), which reduce when $X = x$ to

$$K_\nu(x)I_\mu(x) = \int_0^\infty J_{\mu+\nu}(2x\sinh t)e^{(\nu-\mu)t}\,dt.$$

1.2. Notation

Throughout the work the numbers X, x are to be considered as positive unless a statement to the contrary is made. If y is any positive number, arguments are fixed by the conventions

$$\arg y = 0, \qquad \arg(yi) = \tfrac{1}{2}\pi, \qquad \arg(-yi) = -\tfrac{1}{2}\pi.$$

In order to set out the work as shortly as possible we standardize certain abbreviations (taking the positive square root in all cases):

$$\lambda = \sqrt{(X^2+x^2+2Xx\cos 2\theta)}, \qquad e^{\psi i} = e^{-2\nu\theta i}\left(\frac{X+xe^{2\theta i}}{X+xe^{-2\theta i}}\right)^{\frac{1}{2}(\mu+\nu)},$$

$$\Lambda = \sqrt{(X^2+x^2+2Xx\cosh 2u)}, \qquad e^{w} = e^{2\nu u}\left(\frac{X+xe^{-2u}}{X+xe^{2u}}\right)^{\frac{1}{2}(\mu+\nu)},$$

$$\alpha = \tfrac{1}{2}\log(X/x);$$

when $X > x$ and $1 \leqslant e^{2u} \leqslant X/x$,

$$\Lambda_1 = \sqrt{(X^2+x^2-2Xx\cosh 2u)}, \qquad e^{w_1} = e^{2\nu u}\left(\frac{X-xe^{-2u}}{X-xe^{2u}}\right)^{\frac{1}{2}(\mu+\nu)};$$

when $X > x$ and $e^{2u} \geqslant X/x$,

$$\Lambda_2 = \sqrt{(2Xx\cosh 2u-X^2-x^2)}, \qquad e^{w_2} = e^{2\nu u}\left(\frac{X-xe^{-2u}}{xe^{2u}-X}\right)^{\frac{1}{2}(\mu+\nu)}.$$

* *Quart. J. of Math.* (Oxford), 4 (1933), 193–208. References to this paper are denoted by I.

† Ibid. 1 (1930), 122–45; at 143–5.

2. We begin by noting that the formula I (6.35), wherein

$$\varpi = \sqrt{(X^2+x^2-2Xx\cos\phi)}, \qquad \Omega = \sqrt{(X^2+x^2+2Xx\cosh u)},$$

may be written as

$$2\pi\sin(\mu+\nu)\pi\, J_\mu(X)J_\nu(x) \qquad [X > x,\ R(\mu-\nu) < \tfrac{1}{2}]$$

$$= \int_{-\pi}^{\pi} e^{\nu\phi i}\left(\frac{X-xe^{-\phi i}}{X-xe^{\phi i}}\right)^{\frac{1}{2}(\mu+\nu)}\{\sin\mu\pi\, J_{\mu+\nu}(\varpi)+\sin\nu\pi\, J_{-\mu-\nu}(\varpi)\}\, d\phi -$$

$$-2\sin\nu\pi\int_0^\infty e^{-\nu u}\left(\frac{X+xe^{u}}{X+xe^{-u}}\right)^{\frac{1}{2}(\mu+\nu)}\{\sin\mu\pi\, J_{\mu+\nu}(\Omega)+\sin\nu\pi\, J_{-\mu-\nu}(\Omega)\}\, du.$$

In this form, when $Y_\mu(X)$ replaces $J_\mu(X)$ on the left-hand side, the only changes on the right-hand side are that the functions $Y_{\mu+\nu}$, $Y_{-\mu-\nu}$ replace $J_{\mu+\nu}$, $J_{-\mu-\nu}$: the statement is proved by calculations similar to those whereby I (6.35) was established.

3.1. We next set out the key transformations for our remaining formulae. We assume that $|R(\mu)| < \frac{1}{4}$ and $|R(\nu)| < \frac{1}{4}$, and we leave aside the question of analytic continuation to other values of μ and ν. Further, we suppose that $X > x > 0$ unless the contrary is stated.

Consider

$$\int_0^\infty e^{2\nu U}\left(\frac{X-xe^{-2U}}{X-xe^{2U}}\right)^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}\{\sqrt{(Xe^{U}-xe^{-U})(xe^{U}-Xe^{-U})}\}\, dU, \tag{1}$$

where the path of integration is the real axis indented upwards at the point $U = \alpha$, and the conventions of I (4.41) apply to the multiple-valued functions.

When we increase the argument by $\frac{1}{2}\pi$ and use the substitutions of I, § 5, we see that (1) is equal to*

$$ie^{\nu\pi i}\int_0^{\frac{1}{2}\pi} e^{\psi i}K_{\mu+\nu}(i\lambda)\, d\theta + e^{\nu\pi i}\int_0^\infty e^{w}K_{\mu+\nu}(i\Lambda)\, du. \tag{2}$$

In later stages it is easier to deal with multiples of $\exp(\frac{1}{2}\pi i)$ if we write (1) as

$$\int_0^\alpha e^{w_1}K_{\mu+\nu}(i\Lambda_1)\, du + e^{\frac{1}{2}\pi i(\mu+\nu)}\int_\alpha^\infty e^{w_2}K_{\mu+\nu}(\Lambda_2)\, du. \tag{3}$$

* The result I (5.22) is, in fact, obtained by adding the present result with μ, ν to the corresponding result with $-\mu$, $-\nu$.

The equivalence of (1) and (2) may then be set out as

$$\int_0^\alpha e^{iw_1} K_{\mu+\nu}(i\Lambda_1)\,du + e^{\frac{1}{2}\pi i(\mu+\nu)} \int_\alpha^\infty e^{iw_2} K_{\mu+\nu}(\Lambda_2)\,du$$

$$= ie^{\nu\pi i} \int_0^{\frac{1}{2}\pi} e^{\psi i} K_{\mu+\nu}(i\lambda)\,d\theta + e^{\nu\pi i} \int_0^\infty e^{iw} K_{\mu+\nu}(i\Lambda)\,du. \tag{4}$$

The equation (4) has been proved on the supposition that $X > x > 0$. We may think of it as a formula which involves the two variables X and x/X ($= \xi$, say). Having written it in terms of these two variables and keeping ξ fixed, we find that, by analytic continuation, the equation is true for all X in the half-plane $R(X) > 0$ and that, by continuity, it remains true when $\arg X = -\frac{1}{2}\pi$ (though not when $\arg X = \frac{1}{2}\pi$), provided always that $R(\mu+\nu) < 1$ and $R(\nu-\mu) < \frac{1}{2}$. The latter conditions are satisfied in view of our hypotheses $|R(\mu)| < \frac{1}{4}$, $|R(\nu)| < \frac{1}{4}$.

We may, then, write $X\exp(-\frac{1}{2}\pi i)$ for X and $x\exp(-\frac{1}{2}\pi i)$ for x in (4) and so obtain the formula, valid when $X > x > 0$,

$$\int_0^\alpha e^{iw_1} K_{\mu+\nu}(\Lambda_1)\,du + e^{\frac{1}{2}\pi i(\mu+\nu)} \int_\alpha^\infty e^{iw_2} K_{\mu+\nu}(-i\Lambda_2)\,du$$

$$= ie^{\nu\pi i} \int_0^{\frac{1}{2}\pi} e^{\psi i} K_{\mu+\nu}(\lambda)\,d\theta + e^{\nu\pi i} \int_0^\infty e^{iw} K_{\mu+\nu}(\Lambda)\,du. \tag{5}$$

For brevity we write this formula in the notation

$$\Phi(\mu,\nu,i) = ie^{\nu\pi i} f(\mu,\nu,i) + e^{\nu\pi i} F(\mu,\nu). \tag{6}$$

This is the key formula to much of our later work.

3.2. We may, of course, use the original form (1) and write

$$\Phi(\mu,\nu,i) = \int_0^\infty e^{2\nu u} \left(\frac{X - xe^{-2u}}{X - xe^{2u}}\right)^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}\{\sqrt{(X - xe^{2u})(X - xe^{-2u})}\}\,du,$$

where the arguments of the factors of the integrand are zero for $u = 0$, and at $u = \alpha$ the path of integration has an indentation above the real axis; or in a form more amenable to certain applications,

$$\Phi(\mu,\nu,i) = \tfrac{1}{2} \int_1^\infty t^{\nu-1} \left(\frac{X - x/t}{X - xt}\right)^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}\{\sqrt{(X - x/t)(X - xt)}\}\,dt,$$

where arguments are zero for $t = 1$ and the path of integration has an indentation above the real axis at $t = X/x$.

It follows that $\Phi(\mu, \nu, -i)$ is defined by an integral whose path has an indentation below the real axis at $t = X/x$.

4.1. Identities

The identity (6) of 3.1, namely,

$$\Phi(\mu, \nu, i) = ie^{\nu\pi i}f(\mu, \nu, i)+e^{\nu\pi i}F(\mu, \nu), \tag{1}$$

shows that any product of Bessel functions which can be expressed in terms of the Φ type of integral can also be expressed in terms of the f and F types of integral. This leads to a duplication of formulae which, when encountered in isolated examples, is rather puzzling.

There is a further identity, involving only Φ-integrals. This follows from (1) and the fact that

$$f(\mu, \nu, -i) = f(-\mu, -\nu, i), \tag{2}$$

which is evident since $K_{\mu+\nu}(x) = K_{-\mu-\nu}(x)$. We have, from (1) and its conjugate complex,

$$e^{-\nu\pi i}\Phi(\mu, \nu, i)-e^{\nu\pi i}\Phi(\mu, \nu, -i) = i\{f(\mu, \nu, i)+f(\mu, \nu, -i)\},$$

and this, by (2), is
$$i\{f(\mu, \nu, i)+f(-\mu, -\nu, i)\}.$$

But the latter is unaltered if we write $-\mu$, $-\nu$ for μ, ν, and so

$$e^{-\nu\pi i}\Phi(\mu, \nu, i)-e^{\nu\pi i}\Phi(\mu, \nu, -i)$$
$$= e^{\nu\pi i}\Phi(-\mu, -\nu, i)-e^{-\nu\pi i}\Phi(-\mu, -\nu, -i). \tag{3}$$

This identity is of service in reducing later results to simpler forms.

4.2. Before using the identity (3) we show that it is, essentially, very simple in character. It can, with some labour, be proved as a direct consequence of Cauchy's theorem.

For large values of $|t|$ and for the arguments of $X-xt$, $X-x/t$ which we shall consider,

$$t^{\nu-1}\left(\frac{X-x/t}{X-xt}\right)^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}\{\sqrt{(X-xt)(X-x/t)}\}$$

is at most $O(|t|^{\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{3}{2}}) = o(|t|^{-1})$, and so, by Cauchy's theorem,

$$0 = \int_{\infty}^{(0+)} t^{\nu-1}\left(\frac{X-x/t}{X-xt}\right)^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}\{\sqrt{(X-xt)(X-x/t)}\}\, dt, \tag{1}$$

where the contour has indentations at the points $t = X/x$, $t = x/X$, and
$$\arg t = 0, \qquad \arg(tX-x) = 0, \qquad \arg(X-xt) = -\pi$$
at the starting-point.

This is the identity (3) of 4.1. On taking the parts separately, and using the form given in 3.2, we find

from ∞ to 1 contributes $-2\Phi(\mu,\nu,i)$,
from 1 to 0 contributes $-2\Phi(-\mu,-\nu,-i)$,

and, on the return along the lower side of the real axis,

from 0 to 1 contributes $2e^{2\pi\nu i}\Phi(-\mu,-\nu,i)$,
from 1 to ∞ contributes $2e^{2\pi\nu i}\Phi(\mu,\nu,-i)$.

5. Expressions for $K_\mu(X)K_\nu(x)$, $K_\mu(X)I_\nu(x)$ as integrals of the Φ type; $X > x$

The formula I (4.42) may be written as

$$K_\mu(iX)K_\nu(-ix) = \int\limits_0^\alpha (e^{w_1}+e^{-w_1})K_{\mu+\nu}(i\Lambda_1)\,du +$$

$$+e^{\frac{1}{2}\pi i(\mu+\nu)}\int\limits_\alpha^\infty e^{w_2}K_{\mu+\nu}(\Lambda_2)\,du+e^{-\frac{1}{2}\pi i(\mu+\nu)}\int\limits_\alpha^\infty e^{-w_2}K_{\mu+\nu}(\Lambda_2)\,du.$$

In this we may put $-iX$, $-ix$ for X, x and so obtain

$$K_\mu(X)K_\nu(xe^{-\pi i}) = \Phi(\mu,\nu,i)+\Phi(-\mu,-\nu,i),$$

i.e. $$e^{\nu\pi i}K_\mu(X)K_\nu(x)+\pi iK_\mu(X)I_\nu(x) = \Phi(\mu,\nu,i)+\Phi(-\mu,-\nu,i). \tag{1}$$

From (1) and its conjugate complex we have

$$2\cos\nu\pi\, K_\mu(X)K_\nu(x)$$
$$= \Phi(\mu,\nu,i)+\Phi(-\mu,-\nu,-i)+\Phi(\mu,\nu,-i)+\Phi(-\mu,-\nu,-i), \tag{2}$$

or, on using the identity (3) of 4.1,

$$\begin{aligned} K_\mu(X)K_\nu(x) &= e^{\nu\pi i}\{\Phi(-\mu,-\nu,i)+\Phi(\mu,\nu,-i)\} \\ &= e^{-\nu\pi i}\{\Phi(\mu,\nu,i)+\Phi(-\mu,-\nu,-i)\}. \end{aligned} \tag{3}$$

We also have, from (1) and its conjugate complex,

$$2\pi i\cos\nu\pi\, K_\mu(X)I_\nu(x)$$
$$= e^{-\nu\pi i}\{\Phi(\mu,\nu,i)+\Phi(-\mu,-\nu,i)-e^{\nu\pi i}\{\Phi(\mu,\nu,-i)+\Phi(-\mu,-\nu,-i)\},$$

or, on using the identity (3) of 4.1,

$$\pi i\, K_\mu(X)I_\nu(x) = \Phi(-\mu,-\nu,i)-\Phi(-\mu,-\nu,-i). \tag{4}$$

The equation (4) reduces, almost at once, to the curious formula

$$K_\mu(X)I_\nu(x) = \int\limits_\alpha^\infty e^{-w_2}J_{\mu+\nu}(\Lambda_2)\,du,$$

or, in full,

$$K_\mu(X)I_\nu(x)$$
$$= \int_\alpha^\infty e^{-2\nu u}\left(\frac{X-xe^{-2u}}{xe^{2u}-X}\right)^{-\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}\{\sqrt{(2Xx\cosh 2u-x^2-X^2)}\}\,du. \quad (5)$$

The simplest form of (3) is then seen to be

$$2i\sin\nu\pi\, K_\mu(X)K_\nu(x)$$
$$= \Phi(-\mu,-\nu,-i)-\Phi(-\mu,-\nu,i)+\Phi(\mu,\nu,i)-\Phi(\mu,\nu,-i),$$

or

$$2\sin\nu\pi\, K_\mu(X)K_\nu(x)$$
$$= \pi\int_\alpha^\infty e^{2\nu u}\left(\frac{X-xe^{-2u}}{xe^{2u}-X}\right)^{\frac{1}{2}(\mu+\nu)} J_{-\mu-\nu}\{\sqrt{(2Xx\cosh 2u-x^2-X^2)}\}\,du\, -$$
$$-\pi\int_\alpha^\infty e^{-2\nu u}\left(\frac{X-xe^{-2u}}{xe^{2u}-X}\right)^{-\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}\{\sqrt{(2Xx\cosh 2u-x^2-X^2)}\}\,du. \quad (6)$$

6. The product $I_\mu(X)K_\nu(x);\, X > x$

The task of finding a compact formula for $I_\mu(X)K_\nu(x)$ is by no means a simple one. We begin, as we did for $K_\mu(X)I_\nu(x)$, by quoting I (4.42) in the form

$$K_\mu(iX)K_\nu(-ix) = \int_0^\alpha (e^{w_1}+e^{-w_1})K_{\mu+\nu}(i\Lambda_1)\,du\, +$$
$$+e^{\frac{1}{2}\pi i(\mu+\nu)}\int_\alpha^\infty e^{w_2}K_{\mu+\nu}(\Lambda_2)\,du + e^{-\frac{1}{2}\pi i(\mu+\nu)}\int_\alpha^\infty e^{-w_2}K_{\mu+\nu}(\Lambda_2)\,du. \quad (1)$$

In this we may put iX, ix for X, x and so get

$$K_\mu(Xe^{\pi i})K_\nu(x) = \int_0^\alpha (e^{w_1}+e^{-w_1})K_{\mu+\nu}(\Lambda_1 e^{\pi i})\,du\, +$$
$$+e^{\frac{1}{2}\pi i(\mu+\nu)}\int_\alpha^\infty e^{w_2}K_{\mu+\nu}(i\Lambda_2)\,du + e^{-\frac{1}{2}\pi i(\mu+\nu)}\int_\alpha^\infty e^{-w_2}K_{\mu+\nu}(i\Lambda_2)\,du. \quad (2)$$

We now note that, from the definition of Φ in 3.1 (6),

$$\int_\alpha^\infty e^{w_2}K_{\mu+\nu}(i\Lambda_2)\,du = e^{\frac{1}{2}\pi i(\mu+\nu)}\{\Phi(\mu,\nu,-i)-\int_0^\alpha e^{w_1}K_{\mu+\nu}(\Lambda_1)\,du\},$$

and, on substituting for the α, ∞ integrals, we write (2) as

$$e^{-\mu\pi i}K_\mu(X)K_\nu(x)-\pi i\, I_\mu(X)K_\nu(x)+2\pi i\int_0^\alpha \cosh w_1\, I_{\mu+\nu}(\Lambda_1)\,du$$

$$= e^{-(\mu+\nu)\pi i}\int_0^\alpha (e^{w_1}+e^{-w_1})K_{\mu+\nu}(\Lambda_1)\,du +$$

$$+e^{(\mu+\nu)\pi i}\left\{\Phi(\mu,\nu,-i)-\int_0^\alpha e^{w_1}K_{\mu+\nu}(\Lambda_1)\,du\right\}+$$

$$+e^{-(\mu+\nu)\pi i}\left\{\phi(-\mu,-\nu,-i)-\int_0^\alpha e^{-w_1}K_{\mu+\nu}(\Lambda_1)\,du\right\}$$

$$= -2i\sin(\mu+\nu)\pi\int_0^\alpha e^{w_1}K_{\mu+\nu}(\Lambda_1)\,du +$$

$$+e^{(\mu+\nu)\pi i}\Phi(\mu,\nu,-i)+e^{-(\mu+\nu)\pi i}\Phi(-\mu,-\nu,-i).$$

Now transfer the $K_\mu(X)K_\nu(x)$ term to the other side of the equation and use the formula [§ 5 (3)]

$$K_\mu(X)K_\nu(x) = e^{-\nu\pi i}\{\Phi(\mu,\nu,i)+\Phi(-\mu,-\nu,-i)\}.$$

This gives us

$$-\pi i\, I_\mu(X)K_\nu(x)+2\pi i\int_0^\alpha \cosh w_1\, I_{\mu+\nu}(\Lambda_1)\,du$$

$$= -2i\sin(\mu+\nu)\pi\int_0^\alpha e^{w_1}K_{\mu+\nu}(\Lambda_1)\,du +$$

$$+e^{(\mu+\nu)\pi i}\Phi(\mu,\nu,-i)-e^{-(\mu+\nu)\pi i}\Phi(\mu,\nu,i).$$

When we insert the values of the functions Φ, this becomes

$$e^{\frac{1}{2}(\mu+\nu)\pi i}\int_\alpha^\infty e^{w_2}K_{\mu+\nu}(i\Lambda_2)\,du - e^{-\frac{1}{2}(\mu+\nu)\pi i}\int_\alpha^\infty e^{w_2}K_{\mu+\nu}(-i\Lambda_2)\,du,$$

or, on using I (6.13) and (6.14),

$$-\pi i\int_\alpha^\infty e^{w_2}J_{\mu+\nu}(\Lambda_2)\,du.$$

We have thus proved that

$$I_\mu(X)K_\nu(x) = 2\int_0^\alpha \cosh w_1\, I_{\mu+\nu}(\Lambda_1)\,du + \int_\alpha^\infty e^{w_2}J_{\mu+\nu}(\Lambda_2)\,du. \qquad (3)$$

7. Expressions for $K_\mu(X)K_\nu(x)$, $K_\mu(X)I_\nu(x)$ as integrals of the f and F types; $X > x$

When we make the substitutions

$$\phi(\mu,\nu,i) = e^{\nu\pi i}\{i\,f(\mu,\nu,i)+F(\mu,\nu)\},$$

and so on, in formula (2) of § 5 we find that our result reduces to the integral which formed the starting-point of our investigations in I. Thus our I (2.5), i.e.

$$K_\mu(X)K_\nu(x) = \int_{-\infty}^{\infty} e^{w} K_{\mu+\nu}(\Lambda)\,du,$$

is the f, F form corresponding to the formulae of § 5.

The formula (4) of § 5 becomes

$$\pi i\,K_\mu(X)I_\nu(x) = e^{-\nu\pi i}\{i\,f(-\mu,-\nu,i)+F(-\mu,-\nu)\}-$$
$$-e^{\nu\pi i}\{-i\,f(-\mu,-\nu,-i)+F(-\mu,-\nu)\},$$

and this, with a little simplification, gives the result

$$2\pi\,K_\mu(X)I_\nu(x)$$
$$= \int_{-\pi}^{\pi} e^{-\nu\phi i}\left(\frac{X-xe^{\phi i}}{X-xe^{-\phi i}}\right)^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}\{\sqrt{(X^2+x^2-2Xx\cos\phi)}\}\,d\phi\,-$$
$$-2\sin\nu\pi\int_0^\infty e^{\nu u}\left(\frac{X+xe^{-u}}{X+xe^{u}}\right)^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}\{\sqrt{(X^2+x^2+2Xx\cosh u)}\}\,du.$$

This formula is the analogue of those given in § 2. It can, in fact, be checked by using the formulae of § 2 to derive integrals for

$$2\pi\,K_\mu(-iX)J_\nu(x)e^{-\frac{1}{2}\nu\pi i}.$$

A REDUCIBLE CASE OF THE FOURTH TYPE OF APPELL'S HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES

By W. N. BAILEY (*Manchester*)

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1. As long ago as 1882 Appell* proved the formula

$$F_1(\alpha;\beta,\gamma-\beta;\gamma;X,Y) = (1-Y)^{-\alpha}F\left(\alpha,\beta;\gamma;\frac{X-Y}{1-Y}\right), \qquad (1.1)$$

where $$F_1(\alpha;\beta,\beta';\gamma;X,Y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}\,m!\,n!}X^mY^n,$$

thus showing that the function F_1 can be expressed in terms of an ordinary hypergeometric function when $\gamma = \beta+\beta'$.

No formula of such generality was obtained for the fourth type of function, namely,

$$F_4(\alpha,\beta;\gamma,\gamma';X,Y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n\,m!\,n!}X^mY^n,$$

although Appell showed† that the functions

$$F_4[\alpha,\beta;\gamma,\alpha+\beta-\gamma+1;z^2,(1-z)^2] \quad \text{and} \quad \{F(\alpha,\beta;\gamma;z)\}^2$$

satisfy the same differential equation of the third order.

Some years ago Watson‡ gave the formula

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)F_4[\alpha,\beta;\gamma,\alpha+\beta-\gamma+1;zZ,(1-z)(1-Z)]+\\+\{(1-z)(1-Z)\}^{\gamma-\alpha-\beta}\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)\times\\\times F_4[\gamma-\alpha,\gamma-\beta;\gamma,\gamma-\alpha-\beta+1;zZ,(1-z)(1-Z)]\\= \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)F(\alpha,\beta;\gamma;z)F(\alpha,\beta;\gamma;Z),\end{aligned} \qquad (1.2)$$

and he stated that this appeared to be the best theorem concerning the expression of functions of the fourth type in terms of hypergeometric functions.§

In this paper I give a simpler theorem of this nature, in which only

* Appell (**1**), 175.

† Appell (**2**), 418–21. Appell stated the result explicitly only in the case when $\gamma' = \alpha+\beta-\gamma+1$.

‡ Watson (**5**).

§ Watson also gave a relation connecting four series F_4 with two products of hypergeometric functions, but this can easily be derived from the formula quoted above.

one series F_4 appears. Watson's formula is really a combination of two formulae of the type given in § 2.

2. The theorem is that

$$F_4[\alpha, \beta; \gamma, \alpha+\beta-\gamma+1; z(1-Z), Z(1-z)]$$
$$= F(\alpha, \beta; \gamma; z) F(\alpha, \beta; \alpha+\beta-\gamma+1; Z), \qquad (2.1)$$

this formula being valid inside simply connected regions surrounding $z = 0$, $Z = 0$ for which

$$|z(1-Z)|^{\frac{1}{2}} + |Z(1-z)|^{\frac{1}{2}} < 1.$$

The theorem is true, for example, if $0 < Z < 1$ and $0 < z < 1-Z$.

This result shows that *the function F_4 can be expressed as a product of ordinary hypergeometric functions when $\gamma+\gamma' = \alpha+\beta+1$.*

It will be noticed that, when $Z = 1-z$, the function on the left of (2.1) is that considered by Appell, and the function on the right satisfies the same linear differential equation as that satisfied by $\{F(\alpha, \beta; \gamma; z)\}^2$.

The theorem can be proved by an argument similar to that adopted by Watson, but in this proof it appears to be necessary to use the formula*

$$-\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha_1+s)\Gamma(\alpha_2+s)\Gamma(\alpha_3+s)\Gamma(1-\beta_1-s)\Gamma(-s)\, ds}{\Gamma(\beta_2+s)}$$
$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(1-\beta_1+\alpha_1)\Gamma(1-\beta_1+\alpha_2)\Gamma(1-\beta_1+\alpha_3)}{\Gamma(\beta_2-\alpha_1)\Gamma(\beta_2-\alpha_2)\Gamma(\beta_2-\alpha_3)},$$

where $$\beta_1+\beta_2 = \alpha_1+\alpha_2+\alpha_3+1,$$

instead of the more familiar Barnes's lemma used by Watson. Consequently, I give the more elementary argument which now follows:

$$F_4[\alpha, \beta; \gamma, \alpha+\beta-\gamma+1; z(1-Z), Z(1-z)]$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\alpha+\beta-\gamma+1)_n\, m!\, n!} \{z(1-Z)\}^m \{Z(1-z)\}^n$$
$$= \sum_{N=0}^{\infty} \sum_{m=0}^{N} \frac{(\alpha)_N(\beta)_N}{(\gamma)_m(\alpha+\beta-\gamma+1)_{N-m}\, m!\,(N-m)!} \{z(1-Z)\}^m \{Z(1-z)\}^{N-m}$$
$$= \sum_{N=0}^{\infty} \frac{(\alpha)_N(\beta)_N}{N!(\alpha+\beta-\gamma+1)_N} Z^N (1-z)^N F\left[\begin{matrix}\gamma-\alpha-\beta-N, -N \\ \gamma\end{matrix}; \frac{(z/Z)-z}{1-z}\right]$$

* Barnes (**4**). As usual with integrals of this type, the contour is curved, if necessary, to separate the increasing and decreasing sequences of poles.

$$= \sum_{N=0}^{\infty} \frac{(\alpha)_N(\beta)_N}{N!(\alpha+\beta-\gamma+1)_N} Z^N(1-z)^{\gamma-\alpha-\beta} \times$$
$$\times F_1[\gamma-\alpha-\beta-N; -N, \gamma+N; \gamma; z/Z, z]$$
$$= (1-z)^{\gamma-\alpha-\beta} \sum_{N=0}^{\infty} \sum_{m=0}^{N} \sum_{n=0}^{\infty} \frac{(\alpha)_N(\beta)_N Z^N}{N!(\alpha+\beta-\gamma+1)_N} \times$$
$$\times \frac{(\gamma-\alpha-\beta-N)_{m+n}(-N)_m(\gamma+N)_n}{(\gamma)_{m+n}\, m!\, n!} \left(\frac{z}{Z}\right)^m z^n.$$

Here we have used (1.1). Now put $N = p+m$, $n = q-m$, and we obtain

$$(1-z)^{\gamma-\alpha-\beta} \times$$
$$\times \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{m=0}^{q} \frac{(-1)^p(\alpha)_{p+m}(\beta)_{p+m}(\gamma-\alpha-\beta)_{q-p-m}(\gamma+p+m)_{q-m} Z^p z^q}{p!(\gamma)_q\, m!(q-m)!}$$
$$= (1-z)^{\gamma-\alpha-\beta} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\alpha)_p(\beta)_p(\gamma-\alpha-\beta)_{q-p}(\gamma+p)_q Z^p z^q}{p!\,q!(\gamma)_q} \times$$
$$\times {}_3F_2\left[\begin{matrix}\alpha+p, \beta+p, & -q \\ \gamma+p, & 1+\alpha+\beta-\gamma-q+p\end{matrix}\right].$$

The series ${}_3F_2$ can be summed by Saalschütz's theorem, and we obtain

$$(1-z)^{\gamma-\alpha-\beta} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(\alpha)_p(\beta)_p Z^p}{p!(\alpha+\beta-\gamma+1)_p} \frac{(\gamma-\alpha)_q(\gamma-\beta)_q z^q}{q!(\gamma)_q}$$
$$= (1-z)^{\gamma-\alpha-\beta} F\left[\begin{matrix}\alpha, & \beta; & Z \\ & \alpha+\beta-\gamma+1 & \end{matrix}\right] F\left[\begin{matrix}\gamma-\alpha, \gamma-\beta; z \\ \gamma\end{matrix}\right]$$
$$= F\left[\begin{matrix}\alpha, \beta; z \\ \gamma\end{matrix}\right] F\left[\begin{matrix}\alpha, & \beta; & Z \\ & \alpha+\beta-\gamma+1 & \end{matrix}\right].$$

In this proof we assume that $|z|$ and $|Z|$ are sufficiently small,* and then the complete result follows by analytic continuation.

3. Now in (2.1) change z, Z into $1-Z$, $1-z$, and we find that

$$F_4[\alpha, \beta; \gamma, \alpha+\beta-\gamma+1; z(1-Z), Z(1-z)]$$
$$= F(\alpha, \beta; \gamma; 1-Z) F(\alpha, \beta; \alpha+\beta-\gamma+1; 1-z) \qquad (3.1)$$

inside simply connected regions surrounding $z = 1$, $Z = 1$, for which

$$|Z(1-z)|^{\frac{1}{2}} + |z(1-Z)|^{\frac{1}{2}} < 1.$$

In particular it is true if $0 < Z < 1$ and $1-Z < z < 1$.

This result, together with (2.1), gives the complete expression of

* It is sufficient to assume that $|z|$ and $|Z|$ are each less than $\frac{1}{2}(\sqrt{2}-1)$.

the function F_4 in terms of ordinary hypergeometric functions when $\gamma+\gamma' = \alpha+\beta+1$.

4. It is perhaps worth while to justify the assertion, made at the end of § 1, that Watson's formula can be derived from (2.1). For, using the theorem twice, we see that the left-hand side of (1.2) is

$$\begin{aligned}
&\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)F(\alpha,\beta;\gamma;z)F(\alpha,\beta;\alpha+\beta-\gamma+1;1-Z)+\\
&\qquad+\{(1-z)(1-Z)\}^{\gamma-\alpha-\beta}\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)\times\\
&\qquad\qquad\times F(\gamma-\alpha,\gamma-\beta;\gamma;z)F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-Z)\\
&= F(\alpha,\beta;\gamma;z)[\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)F(\alpha,\beta;\alpha+\beta-\gamma+1;1-Z)+\\
&\qquad+\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)\times\\
&\qquad\qquad\times(1-Z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-Z)]\\
&= F(\alpha,\beta;\gamma;z)\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)F(\alpha,\beta;\gamma;Z).
\end{aligned}$$

This argument assumes that z, Z lie in certain regions surrounding $z = 0$, $Z = 1$, but the symmetry of the result shows that it is also true in the regions surrounding $z = 1$, $Z = 0$.

5. Finally I mention that the present investigation of (2.1) was suggested by the formula*

$$\begin{aligned}
&F(\alpha,\beta;\gamma;z)F(\alpha,\beta;\alpha+\beta-\gamma+1;z)\\
&\qquad= {}_4F_3\left[\begin{matrix}\alpha,\beta, & \tfrac{1}{2}(\alpha+\beta), & \tfrac{1}{2}(\alpha+\beta+1);4z(1-z)\\ \gamma,\alpha+\beta-\gamma+1, & \alpha+\beta & \end{matrix}\right].
\end{aligned}$$

The reader will readily see the connexion between this result and the special case of (2.1) in which $Z = z$. It can, in fact, be derived from (2.1) by a use of Vandermonde's theorem.

* Bailey (**3**), § 6.

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ON A PROBLEM IN THE ADDITIVE THEORY OF NUMBERS (VI)

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1. Introductory

In previous papers we discussed the representation of a large number n as the sum of s Nth power-free numbers or M-numbers, that is, of s numbers not divisible by an Nth power greater than 1. Here N is fixed once and for all, repetitions are allowed, and order is regarded as relevant. Using an elementary argument, we were able to show† that for $N \geqslant 2$, $s \geqslant 2$ the number of representations

$$\nu_s(n) = \frac{n^{s-1}}{(s-1)!}\frac{1}{\zeta^s(N)} S(N,s;n)\Big\{1+O\Big(n^{-1+\frac{2}{N+1}+\epsilon}\Big)\Big\} \tag{1.1}$$

as $n \to \infty$, where the factor $S(N,s;n)$ lies between two positive absolute constants; the argument was later simplified by Estermann.‡ Using analytic function theory, on the other hand, we obtained§ for $N \geqslant 2$, $s \geqslant 3$ the formula

$$\nu_s(n) = \frac{n^{s-1}}{(s-1)!}\frac{1}{\zeta^s(N)} S(N,s;n)\Big\{1+O\Big(n^{-1+\frac{1}{N}+\frac{N-1}{Ns}+\epsilon}\Big)\Big\} \tag{1.2}$$

which is an improvement on (1.1) when $s \geqslant N+2$.

In the present paper we extend these results to the case in which the M-numbers all belong to a given arithmetic progression, using an adaptation of Estermann's argument, followed by an induction, to prove

Theorem A_1. *Let $s \geqslant 2$, $N \geqslant 2$, $1 \leqslant b \leqslant a$; let $a = \prod_{i=1}^{\infty} p_i^{\alpha_i}$ be the expression of a as a product of primes in ascending order; let (a, b) be Nth power-free,‖ and let*

$$S(n) = S(a,b,N,s;n)$$

$$= \prod_{p_i^N \nmid n,\, p_i \nmid \frac{a}{(a,b)}} \left(1+\frac{(-1)^{s+1}}{(p_i^{N-\alpha_i}-1)^s}\right) \prod_{p_i^N \mid n,\, p_i \mid \frac{a}{(a,b)}} \left(1+\frac{(-1)^s}{(p_i^{N-\alpha_i}-1)^{s-1}}\right).$$

† Evelyn and Linfoot (**2**) (II), (III) (see list of references).

‡ Estermann (**1**).

§ Evelyn and Linfoot (**2**) (V).

‖ Unless (a, b) is Nth power-free, the progression (1.3) evidently cannot contain an M-number.

Then the number of representations of a large number n as the sum of s M-numbers, each belonging to the arithmetic progression

$$b, b+a, b+2a, \ldots, \tag{1.3}$$

repetitions being allowed and order being regarded as relevant, is

$$\nu_s(n) = \frac{1}{(s-1)!}\left(\frac{n}{a}\right)^{s-1} \prod_{p_i \nmid \frac{a}{(a,b)}} \left(1-\frac{1}{p_i^{N-\alpha_i}}\right)^s S(n)\left\{1+O\left(n^{-1+\frac{2}{N+1}+\epsilon}\right)\right\} \tag{1.4}$$

for every $\epsilon > 0$, *if* $n \equiv sb \pmod a$, *while for* $n \not\equiv sb \pmod a$

$$\nu_s(n) = 0.$$

By using the Winogradoff-Gelbcke argument of (**2**) (V) we are able to obtain the (sometimes sharper)

THEOREM A_2. *For* $s \geqslant 3$, $N \geqslant 2$ *and with the notation of Theorem* A_1

$$\nu_s(n) = \frac{1}{(s-1)!}\left(\frac{n}{a}\right)^{s-1} \prod_{p_i \nmid \frac{a}{(a,b)}} \left(1-\frac{1}{p_i^{N-\alpha_i}}\right)^s S(n)\left\{1+O\left(n^{-1+\frac{1}{N}+\frac{N-1}{Ns}+\epsilon}\right)\right\} \tag{1.5}$$

if $n \equiv sb \pmod a$, *while for* $n \not\equiv sb \pmod a$

$$\nu_s(n) = 0.$$

As the proof is very similar to that for the case $a = 1$ we do not reproduce it.

To interpret the expression

$$\frac{1}{(s-1)!}\left(\frac{n}{a}\right)^{s-1} \prod_{p_i \mid \frac{a}{(a,b)}} \left(1-\frac{1}{p_i^{N-\alpha_i}}\right)^s S(n)$$

we observe that the number of representations of n as the sum of s positive integers of the form $ar+b$ is asymptotic to $\frac{1}{(s-1)!}\left(\frac{n}{a}\right)^{s-1}$ as $n \to \infty$ through values congruent to $sb \pmod a$. To pass from this to the representation by M-numbers we naturally introduce a factor

$$\prod_{p_i \nmid \frac{a}{(a,b)}} \left(1-\frac{1}{p_i^{N-\alpha_i}}\right)^s,$$

since (see Lemma 3.81) the density of M-numbers in (1.3) is

$$\frac{1}{a} \prod_{p_i \nmid \frac{a}{(a,b)}} \left(1-\frac{1}{p_i^{N-\alpha_i}}\right).$$

Finally there will be a certain interference, depending on the form

of n, through the restriction imposed by the equation

$$M_1+M_2+\ldots+M_s = n$$

on the arithmetic structure of the M's; this corresponds to the factor $S(n)$.

Just as in the special case $a = 1$ of (I), we can infer at once from our asymptotic formula

THEOREM B. *The number of partitions of a large Nth power-free number M into s Nth power-free numbers of the form ar+b, where (a, b) is Nth power-free and $M = sb \pmod a$ differs asymptotically from the 'normal' number*

$$\frac{1}{(s-1)!}\left(\frac{M}{a}\right)^{s-1} \prod_{p_i \nmid \frac{a}{(a,b)}} \left(1-\frac{1}{p_i^{N-\alpha_i}}\right)^s$$

by a constant factor $\gamma(s, N, a, b)$ which is less than one when s is even, greater than one when s is odd.

2. We first suppose $s = 2$. The numbers under the summation signs have the following meanings:

1. $x^N u+y^N v = n$; 2. $x^N u \equiv b \pmod a$; 3. $y^N v \equiv b \pmod a$;

4. $\left(x, \frac{a}{c}\right) = 1$; 5. $\left(y, \frac{a}{c}\right) = 1$; 7. $\mu(x), \mu(y) \neq 0$; 8. $x, y \leqslant n^{\frac{2}{N+1}}$;

9. $(x, y)^{N-*} \Big| \frac{n}{c}$; 10. $(x, y)^N | n$.

Since

$$\sum_{m^N | n} \mu(m) = 1 \text{ if } n \text{ is an } M\text{-number}$$
$$= 0 \text{ otherwise,}$$

the number of representations

$$\nu_2(n) = \sum_{\substack{M_1+M_2=n \\ M_i \equiv b \pmod a}} 1 = \sum_{1.2.3.}^{x,y,u,v} \mu(x)\mu(y)$$

$$= \sum^{x,y} \mu(x)\mu(y) \sum_{1.2.3.}^{u,v} 1. \tag{2.1}$$

Let $c = (a, b) = \prod_{i=1}^{\infty} p_i^{\sigma_i}$, where every $\sigma_i \leqslant N-1$, and write $(c, x^N) = \delta$. Then c/δ is prime to x and the congruence

$$x^N u \equiv b \pmod a \tag{2.2}$$

shows that $c | x^N u$, $(c/\delta) | u$. It can therefore be written

$$\frac{x^N}{\delta} \frac{u}{c\delta^{-1}} \equiv b_1 \pmod{a_1}, \tag{2.3}$$

where $a = ca_1$, $b = cb_1$, $(a_1, b_1) = 1$. This has solutions in u only if x^N/δ is prime to a_1; it is then (writing $u_1 = \delta u/c$)

$$u_1 \equiv b_1 \operatorname{soc}_{a_1} \frac{x^N}{c, x^N} \pmod{a_1}, \tag{2.4}$$

where it is immaterial which value of the socius is taken. Let $u_{1,0}$ be the least positive solution in u_1 of this congruence; thus $1 \leqslant u_{1,0} \leqslant a_1$. Then the general solution is

$$u_1 = u_{1,0} + ra_1 \qquad (r \geqslant 0).$$

Thus (2.2) has no solution in u unless

$$\left(\frac{x^N}{(x^N, c)}, \frac{a}{c}\right) = 1,$$

i.e. unless

$$\left(x, \frac{a}{c}\right) = 1;$$

when this is satisfied its general solution is

$$u = \frac{c}{(c, x^N)} u_{1,0} + \frac{ra}{(c, x^N)}.$$

A similar result holds for v. (2.1) therefore becomes

$$\sum_{4.5.7.}^{x,y} \mu(x)\mu(y) \sum_{6.}^{r,s} 1,$$

where 6. stands for

$$a\left(\frac{x^N r}{(c, x^N)} + \frac{y^N s}{(c, y^N)}\right) = n - \frac{cx^N}{(c, x^N)} u_{1,0} - \frac{cy^N}{(c, y^N)} v_{1,0}.$$

We now introduce a new notation. Let $m = \prod_{i=1}^{\infty} p_i^{\mu_i}$ be any positive integer. Then by m^{N-*} we shall mean the integer $\prod_{i=1}^{\infty} p_i^{\mu_i(N-\sigma_i)}$. Since x, y are quadratfrei, condition 6. can be written

$$a(x^{N-*}r + y^{N-*}s) = n - cx^{N-*}u_{1,0} - cy^{N-*}v_{1,0}.$$

This has a solution in r, s only if

$$\frac{a}{c}(x^{N-*}, y^{N-*}) \,\Big|\, \frac{n}{c} - x^{N-*}u_{1,0} - y^{N-*}v_{1,0},$$

which is equivalent to the pair of conditions

$$a \mid n - 2b, \qquad (x^{N-*}, y^{N-*}) \,\Big|\, \frac{n}{c}$$

since a/c is prime to $(x^{N-*}, y^{N-*}) = (x, y)^{N-*}$ and $x^{N-*}u_{1,0}$, $y^{N-*}v_{1,0}$ are each congruent to $b_1 \pmod{a/c}$. Thus $\nu_2(n) = 0$ unless these are

satisfied; when they hold,

$$\sum_{6.}^{r,s} 1 = \frac{(n - x^{N-*}u_{1,0} - y^{N-*}v_{1,0})a(x,y)^{N-*}}{ax^{N-*}\,ay^{N-*}} + 2\theta_1$$

by Lemma 2.1 of (II), where θ denotes generally a number absolutely less than or equal to 1,

$$= \frac{n}{a}\,\frac{(x,y)^{N-*}}{x^{N-*}y^{N-*}} + 4\theta_2$$

since $u_{1,0}$, $v_{1,0} \leqslant a_1$.

Let $$Q_1 = \sum_{4.5.7.8.}^{x,y} \mu(x)\mu(y) \sum_{6.}^{r,s} 1;$$

then, just as in Estermann's original argument,

$$|\nu_2(n) - Q_1| = O\Big(n^{\frac{2}{N+1}+\epsilon}\Big),$$

while now $$Q_1 = \sum_{4.5.7.8.9.} u(x)\mu(y)\Big(\frac{n}{a}\,\frac{(x,y)^{N-*}}{x^{N-*}y^{N-*}} + 4\theta_2\Big)$$

when $a|n-2b$, zero otherwise. The condition 9. $(x,y)^{N-*}\Big|\frac{n}{c}$ is equivalent to 10. $(x,y)^N|n$. Thus, in virtue of 8.,

$$Q_1 = \frac{n}{a} \sum_{4.5.7.8.10.} \mu(x)\mu(y)\frac{(x,y)^{N-*}}{x^{N-*}y^{N-*}} + O\Big(n^{\frac{2}{N+1}}\Big),$$

when $a|n-2b$. We next drop the condition 8. $x, y \leqslant n^{1/N+1}$ in the first term; the error caused is $O(n^{2/(N+1)})$ as with Estermann. Thus for $a|n-2b$

$$\nu_2(n) = \frac{n}{a} \sum_{4.5.7.10.}^{x,y} \mu(x)\mu(y)\frac{(x,y)^{N-*}}{x^{N-*}y^{N-*}} + O\Big(n^{\frac{2}{N+1}+\epsilon}\Big),$$

while $\nu_2(n) = 0$ if $a \nmid n-2b$.

The coefficient is easily evaluated; by a calculation similar to that of (2) (III) we readily verify that for $s \geqslant 1$

$$\sum_{\substack{(x_1,\ldots,x_s,\frac{a}{c})=1 \\ (x_1,\ldots,x_s)^N|n}}^{x_1,\ldots,x_s} \frac{\mu(x_1)}{x_1^{N-*}}\cdots\frac{\mu(x_s)}{x_s^{N-*}}(x_1,\ldots,x_s)^{N-*}$$

$$= \prod_{p_i \nmid \frac{a}{c}} \Big(1 - \frac{1}{p_i^{N-\alpha_i}}\Big)^s \prod_{p_i^N \nmid n,\, p_i \nmid \frac{a}{c}} \Big(1 + \frac{(-1)^{s+1}}{(p_i^{N-\alpha_i}-1)^s}\Big) \prod_{p_i^N | n,\, p_i \nmid \frac{a}{c}} \Big(1 + \frac{(-1)^s}{(p_i^{N-\alpha_i}-1)^{s-1}}\Big)$$

$$= \prod_{p_i \nmid \frac{a}{c}} \Big(1 - \frac{1}{p_i^{N-\alpha_i}}\Big)^s S(a,b,s,N;n). \tag{2.5}$$

Taking $s = 2$, we obtain Theorem A_1 for the case $s = 2$.

Next let $s \geqslant 3$ and suppose the theorem proved for $\nu_{s-1}(n)$. Thus, if $a \mid n-(s-1)b$,

$$\nu_{s-1} = \frac{1}{(s-2)!}\left(\frac{n}{a}\right)^{s-2} \sum_{\substack{\left(x_1,\dots,x_{s-1},\frac{a}{c}\right)=1 \\ (x_1,\dots,x_{s-1})^N \mid n}}^{x_1,\dots,x_{s-1}} \frac{\mu(x_1)}{x_1^{N-*}} \cdots \frac{\mu(x_{s-1})}{x_{s+1}^{N-*}} (x_1,\dots,x_{s-1})^{N-*} + \\ + O\left(n^{s-3+\frac{2}{N+1}+\epsilon}\right)$$

for every $\epsilon > 0$, while if $a \mid n-(s-1)b$, $\nu_{s-1}(n) = 0$. Whence

$$\nu_s(n) = \sum_{\substack{M_1+\dots+M_s=n \\ M_i \equiv b (\operatorname{mod} a)}} 1 = \sum_{\substack{M_s=1 \\ M_s \equiv b (\operatorname{mod} a)}}^{n} \nu_{s-1}(n-M_s)$$

$$= \sum_{\substack{M_s=1 \\ M \equiv b (\operatorname{mod} a)}}^{n} \frac{(n-M_s)^{s-2}}{(s-2)!\,a^{s-2}} \sum_{\substack{\left(x_1,\dots,x_{s-1},\frac{a}{c}\right)=1 \\ (x_1,\dots,x_{s-1})^N \mid n-M_s}}^{x_1,\dots,x_{s-1}} \frac{\mu(x_1)}{x_1^{N-*}} \cdots \frac{\mu(x_{s-1})}{x_{s-1}^{N-*}} (x_1,\dots,x_{s-1})^{N-*} + \\ + O \sum_{\substack{M_s=1 \\ M_s \equiv b (\operatorname{mod} a)}}^{n} (n-M_s)^{s-3+\frac{2}{N+1}+\epsilon},$$

provided $a \mid n-M_s-(s-1)b$, i.e. $a \mid n-sb$, while $\nu_s(n) = 0$ if $a \mid n-sb$. In the former case the rest of the argument is similar to that on pp. 641–4 of (**2**) (III); we therefore merely state its result, namely, that for $a \mid n-sb$

$$\nu_s(n) = \frac{1}{(s-1)!}\left(\frac{n}{a}\right)^{s-1} \sum_{\substack{\left(x_1,\dots,x_{s-1},\frac{a}{c}\right)=1 \\ (x_1,\dots,x_s)^N \mid n}}^{x_1,\dots,x_s} \frac{\mu(x_1)}{x_1^{N-*}} \cdots \frac{\mu(x_s)}{x_s^{N-*}} (x_1,\dots,x_s)^{N-*} + \\ + O\left(n^{s-2+\frac{2}{N+1}+\epsilon}\right),$$

while for $a \nmid n-sb$, $\nu_s(n) = 0$. In virtue of (2.5), this proves Theorem A_1 by induction.†

† On p. 641, l. 16, of (**2**) (III) an error term $O\left(n^{s-2+\frac{2}{N+1}+\epsilon}\right)$ is omitted.

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ON A MEAN FREE PATH FORMULA

By E. A. MILNE (*Oxford*)

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1. THE following note arose out of the recent brilliant work of Blackett and Occhialini* disclosing the probability of the existence of a positive electron. Following the ideas of Dirac, Blackett and Occhialini evaluated the probability that a positive electron will disappear by reacting with a negative electron to form two photons during a length of path in which its energy, if it had not been annihilated, would have fallen from E_1 to E_2. A formula is obtained below which relates this probability to the mean free path for annihilation. The analysis differs from the usual analysis in the kinetic theory of gases owing to the fact that the future mean free path of a particle is a function of its present energy, and so varies along the path; in the kinetic theory of gases the future mean free path for collision (for a particle of constant velocity) is independent of the portion of path already traversed.

2. Let $f(E)\epsilon$ be the probability that a particle is annihilated whilst its energy, if it had not been annihilated, would have fallen from E to $E-\epsilon$ (ϵ small and positive). Let dr be the corresponding length of path for non-annihilation, and write

$$\frac{dr}{dE} = -g(E) \qquad (g(E) > 0). \tag{1}$$

Let $\phi(E_1, E)$ be the probability that a particle *avoids* annihilation whilst its energy falls from E_1 to E, r being the corresponding length of path described.

Then if N particles start with energy E_1, $N\phi(E_1, E)$ survive to energy E, and $N\phi(E_1, E-\epsilon)$ to energy $E-\epsilon$. The loss of particles between E and $E-\epsilon$ is accordingly

$$N\frac{\partial\phi(E_1, E)}{\partial E}\epsilon,$$

and this must be equal to

$$N\phi(E_1, E)f(E)\epsilon.$$

Hence
$$\frac{\partial\phi(E_1, E)}{\partial E} = \phi(E_1, E)f(E),$$

* *Proc. Roy. Soc.* A, 139 (1933), 699.

or
$$\log\phi(E_1, E) = -\int_E^{E_1} f(E)\,dE \qquad (E < E_1). \tag{2}$$

The particles annihilated in the interval $(E, E-\epsilon)$ have a range r. Hence the mean range, or mean free path for annihilation, of particles starting with energy E_1, say $\lambda(E_1)$, is given by
$$\lambda(E_1) = \int_{E=0}^{E=E_1} \frac{\partial\phi(E_1, E)}{\partial E} r\,dE.$$
Integrating by parts and noting that
$$\text{at } E = E_1, \qquad r = 0,$$
$$\text{at } E = 0, \qquad \phi(E_1, E) = 0$$
(assuming that all the particles are annihilated by the time $E = 0$), we have
$$\lambda(E_1) = -\int_0^{E_1} \phi(E_1, E)\frac{dr}{dE}\,dE$$
$$= \int_0^{E_1} \phi(E_1, E)g(E)\,dE, \tag{3}$$
by (1). Hence, by (2),
$$\lambda(E_1) = \int_0^{E_1} e^{-\int_E^{E_1} f(E)\,dE} g(E)\,dE. \tag{4}$$

We now differentiate with respect to E_1. The details are most conveniently carried out by writing
$$f(E) = F'(E), \qquad \int_E^{E_1} f(E)\,dE = F(E_1)-F(E).$$
We find
$$\frac{d\lambda(E_1)}{dE_1} = g(E_1)-f(E_1)\int_0^{E_1} e^{-\int_E^{E_1} f(E)\,dE} g(E)\,dE$$
$$= g(E_1)-f(E_1)\lambda(E_1), \tag{5}$$
by (4); (5) is the essential formula. Omitting now the suffix 1, we have
$$f(E) = \frac{g(E)}{\lambda(E)}-\frac{1}{\lambda(E)}\frac{d\lambda(E)}{dE}$$
$$= -\frac{1}{\lambda(E)}\frac{dr}{dE}-\frac{1}{\lambda(E)}\frac{d\lambda(E)}{dE}. \tag{6}$$

Inserting in (2) we find

$$\log \phi(E_1, E) = \int\limits_{r=r(E_1,E)}^{r=0} \frac{dr}{\lambda(E)} + \log\frac{\lambda(E_1)}{\lambda(E)}, \tag{7}$$

or

$$\log \phi(E_1, E_2) = -\int\limits_{E=E_1, r=0}^{E=E_2, r=r(E_1,E_2)} \frac{dr}{\lambda(E)} + \log\frac{\lambda(E_1)}{\lambda(E_2)}. \tag{8}$$

In Blackett and Occhialini's notation,

$$\phi(E_1, E_2) = 1 - \Phi(E_1, E_2),$$

so that (8) may be written

$$\log[1-\Phi(E_1, E_2)] = -\int\limits_{E_1}^{E_2} \frac{dr}{\lambda(E)} + \log\frac{\lambda(E_1)}{\lambda(E_2)}. \tag{9}$$

This is the desired formula.

3. Introduce now a new variable R defined by

$$r(E_1, E) = R(E_1, E) - \lambda(E), \tag{10}$$

so that*

$$dr = dR - \lambda'(E)\,dE. \tag{10'}$$

Then (9) reduces to

$$\log[1-\Phi(E_1, E_2)] = -\int\limits_{E_1}^{E_2} \frac{dR}{\lambda(E)}, \tag{11}$$

where

$$\text{at } E = E_1, \qquad R = \lambda(E_1),$$
$$\text{at } E = E_2, \qquad R = \lambda(E_2) + r(E_1, E_2).$$

The physical meaning of $R(E_1, E)$ is that it is the total path which a particle would describe if it (*a*) escaped annihilation whilst decreasing in energy from E_1 to E, (*b*) then pursued a path equal to the mean free path for annihilation from energy E. It may be noted from (10′) that dR is not equal to the element of path dr unless $\lambda'(E) = 0$, i.e. unless the future mean free path for annihilation is independent of the energy.

4. In the elementary treatment of mean free paths in gas-theory, the chance of a collision within a further distance dr is simply dr/λ, where λ is independent of the distance already traversed. This agrees with (5) on taking $\lambda(E)$ independent of E, since the chance in question is $-f(E)dE$, i.e. $-g(E)dE/\lambda$, i.e. dr/λ. The factor $\lambda(E_1)/\lambda(E)$ in (9) then, of course, reduces to unity.

* dR depends only on E and dE.

5. Writing (8) in the form

$$\phi(E_1, E_2) = \frac{\lambda(E_1)}{\lambda(E_2)} e^{-\int_{E_1}^{E_2} \frac{dr}{\lambda(E)}}, \tag{12}$$

we readily verify that the right-hand side of (12) is necessarily less than unity, for $E_2 < E_1$. Formula (12) is fully analogous to the formula for the density ρ at height r in a non-isothermal atmosphere, namely,

$$\frac{\rho}{\rho_0} = \frac{T_0}{T} e^{-\frac{mg}{k}\int \frac{dr}{T}},$$

$\lambda(E)$ playing the part of temperature T.

6. I am indebted to Dr. Blackett and Professor Dirac for pointing out that in the formula used by Blackett and Occhialini,* namely,

$$\log[1-\Phi(E_1, E_2)] = -\int_{E_2}^{E_1} \frac{dr}{\Lambda(E)}, \tag{13}$$

$\Lambda(E)$ is not the actual mean free path of a particle of initial energy E but the mean free path for annihilation assuming the particle continued with constant energy E, as would be the case, for example, if a suitable external field were applied; (13) is equivalent to (2), since $-f(E)\,dE$ is here dr/Λ. It follows by comparison of (13) and (9), or by direct calculation of $\Lambda(E)$, that

$$1-\frac{\lambda(E)}{\Lambda(E)} = -\lambda'(E)\frac{dE}{dr}, \tag{14}$$

which is the relation between $\Lambda(E)$ and $\lambda(E)$. Naturally always $\Lambda(E) > \lambda(E)$. In Blackett and Occhialini's paper r and $\Lambda(E)$ are tabulated numerically as functions of E, and Φ is computed from (13).

* Loc. cit. 715. A misprint of sign has been corrected.

THE UNIQUE FACTORIZATION OF A POSITIVE INTEGER

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THERE are several demonstrations of the theorem that it is not possible to split up a number into prime factors in various different ways. The following very brief and simple proof appears not previously to have been published.

The argument rests upon the fact that it may be shown that if there were any number which could be resolved into primes in more than one way, one could always find a smaller number of which the same was true. By repeating this process, we obtain a decreasing sequence of positive integers each of which can be resolved into primes in more than one way. But such a sequence must eventually contain a prime or unity, for either of which decomposition is unique. We have then a contradiction.

Suppose there is a number n which has two different compositions

$$n = p_1 p_2 p_3 \dots = q_1 q_2 q_3 q_4 \dots .$$

We wish to show that in this case there must be a smaller number n' with the same property.

We may suppose that none of the factors p_1, p_2, p_3 is identical with any of the factors q_1, q_2, q_3,..., for obviously if any q were equal to any p, one could cancel on both sides and thus obtain a number n' smaller than the original number n having two different decompositions. Further, we may suppose that p_1 is the smallest of the p's and q_1 the least of the q's. Then one of these must be $\leqslant \sqrt{n}$ and the other $< \sqrt{n}$, so that $p_1 q_1 < n$.

Hence $$p_1 p_2 p_3 \dots - p_1 q_1 = q_1 q_2 q_3 q_4 \dots - p_1 q_1$$
is a positive number and

$$p_1(p_2 p_3 \dots - q_1) = q_1(q_2 q_3 q_4 \dots - p_1).$$

There are two possibilities:

(*a*) q_1 divides $p_2 p_3 \dots$.

Then $n' = p_2 p_3 \dots$, smaller by the factor p_1 than n, has two decompositions, one of which contains q_1 whilst the other does not.

(*b*) q_1 does not divide $p_2 p_3 \ldots$.

Then it does not divide $p_2 p_3 \ldots - q_1$.

Hence $$n' = p_1(p_2 p_3 \ldots - q_1) = p_1 \alpha_2 \alpha_3 \ldots,$$
where no α is q_1, whilst since
$$n' = q_1(q_2 q_3 q_4 \ldots - p_1)$$
it also has a decomposition in which one prime *is* q_1. Thus n', which is smaller by $p_1 q_1$ than n, has two different decompositions.

Since the same reasoning can be applied to the number n' as to n, the process can be repeated as often as one likes, so that if any number could be split up into prime factors in two different ways, it would be possible to construct numbers as small as one pleased, which had the same property.

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